Transport of parallel momentum by drift-Alfvén turbulence

C. J. McDevitt and P. H. Diamond

*Center for Astrophysics and Space Sciences and Department of Physics, University of California
at San Diego, La Jolla, California 92093-0424, USA*

An electromagnetic gyrokinetic formulation is utilized to calculate the turbulent radial flux of parallel momentum for a strongly magnetized plasma in the large aspect ratio limit. For low-β plasmas, excluding regions of steep density gradients, the level of momentum transport induced by microturbulence is found to be well described within the electrostatic approximation. However, near regions of steep equilibrium profile gradients, strong electromagnetic contributions to the momentum flux are predicted. In particular, for sufficiently steep density gradient, the magnitude of transport induced by the off-diagonal residual stress component of the momentum flux induced by drift wave turbulence can be quenched. This quenching mechanism, which results from shielding of the parallel electric field by the inductive term, is distinct from $\mathbf{E} \times \mathbf{B}$ shear decorrelation, since it allows for the level of off-diagonal turbulent transport to be strongly reduced without extinguishing the underlying microturbulence. In contrast, the level of transport induced by a given Alfvénic branch of the drift-Alfvén dispersion relationship typically increases as the density gradient steepens, allowing an alternate channel for momentum transport. A calculation of the momentum transport induced by Alfvénic turbulence in a homogeneous medium suggests that an imbalance in Elsasser populations is required in order to introduce a finite level of off-diagonal momentum transport for the case of the simplified geometry considered. © *2009 American Institute of Physics.*
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I. INTRODUCTION

The observation of significant rotation in the absence of external momentum sources, often referred to as "spontaneous rotation," has proven a welcome surprise to the magnetic fusion program.[1] A compilation from a broad database of H-mode plasmas, has shown that this "spontaneous" or "intrinsic" rate of rotation is proportional to the stored plasma energy divided by the plasma current. Extrapolation to ITER relevant parameters, based on the ITPA database alluded to above, suggests that this intrinsic rate of rotation is likely to be sufficient to suppress resistive wall modes for ITER plasmas.[2] While this optimistic prediction is encouraging, a more detailed understanding of the underlying physical processes is clearly desirable. Specifically, multiple theoretical studies have suggested that strong off-diagonal components to the momentum flux may be induced by small scale microturbulence. These off-diagonal transport terms can be separated into contributions which are independent of both the toroidal flow velocity and its gradient, which we will refer to as residual stress terms, and convective contributions which are proportional to the toroidal flow velocity. The latter of these two off-diagonal contributions arises either due to the breaking of translational invariance along magnetic field lines, or convective transport of mean parallel momentum by particle fluxes. The first of these two mechanisms is closely associated with the equilibrium magnetic field topology,[3–5] whereas the second depends sensitively on the electron response. Terms of this form have been shown to play a key role in describing rotation profile peaking,[6] or conversely, the formation of hollow rotation profiles.

The residual stress on the other hand, which corresponds to the portion of the momentum flux which is independent of both the toroidal flow velocity and its gradient, provides a natural candidate for driving intrinsic rotation. This can easily be seen by considering the structure of the momentum flux in a simplified geometry, i.e.,

$$\frac{\partial \langle v_\phi \rangle}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\chi_\phi \frac{\partial \langle v_\phi \rangle}{\partial r} + V_c \langle v_\phi \rangle + S\right)\right] = S_{\text{ext}}.$$

Assuming stationary solutions, and integrating from 0 to r, this expression may be written as

$$-\chi_\phi \frac{\partial \langle v_\phi \rangle(r)}{\partial r} + V_c \langle v_\phi \rangle(r) + S(r) = \frac{1}{r}\int_0^r dr' r' S_{\text{ext}}(r')$$
$$= \hat{S}_{\text{ext}}(r),$$

where $\hat{S}_{\text{ext}}$ is the integrated external momentum source. Hence, while clearly physically distinct, the residual stress has a mathematical form which is isomorphic to the integrated momentum source. Thus, the residual stress corresponds to an ideal candidate for understanding offsets in the toroidal rotation velocity. This is in contrast to the convective or "pinch" term, whose magnitude is dependent on the local rotation velocity, and thus has a fundamentally distinct effect on rotation profiles. Specifically, systems whose only nondiffusive contribution to the momentum flux is convective, do not admit stationary solutions with a finite rate of rotation without the presence of either an external momentum source, or a nonzero edge boundary condition.

Aside from having a mathematically distinct form in comparison to convective terms, the residual stress has a manifestly different physical origin. Residual stress terms are

often predicted to arise due to $k_\parallel$ symmetry breaking induced by the presence of radial electric field shear.[7,8] While this mechanism does not appear to be a unique means of introducing a nonvanishing residual stress term (see Refs. 9 and 10 for alternate mechanisms), it is likely to be the most robust mechanism in H-mode plasmas, especially near the plasma edge. This follows since while this portion of the residual stress is being enhanced by radial electric field shear, the remaining terms in the momentum flux are reduced due to $\mathbf{E} \times \mathbf{B}$ shear decorrelation.[11,12] Thus, the radial electric field shear driven portion of the residual stress is likely to play a conspicuous role in barrier regions.[7,13] This property is particularly suggestive, since the "rice scaling" inferred from the ITPA database is only valid in H-mode plasmas, where strong radial electric field shear is present near the plasma edge. Taken together, these two observations suggest that the structure of the plasma edge, along with the radial electric field shear driven portion of the residual stress are likely to be key components in the description of intrinsic rotation.[13]

The link between radial electric field shear and intrinsic rotation has been further strengthened by perturbation experiments carried out on JT-60U.[14] In these experiments, for plasmas with weak pressure gradients, the calculated value of rotation based on a transient momentum transport analysis, in which a diffusive and convective component of the momentum flux are computed, are in good agreement with the measured rotation profile. However, as the pressure gradient is steepened in the core, a large offset in the rate of rotation is observed. This offset in rotation correlates well with the region of strong pressure gradient, and hence, from radial force balance, the region of strong radial electric field shear, thus providing an additional link between rotation offsets and strong radial electric field shear.

While existing theoretical models of parallel momentum transport suggest that robust rates of intrinsic rotation and radial electric field shear are closely linked, turbulence models analyzed up to this point have been purely electrostatic. The electrostatic approximation is particularly conspicuous in this context, since regions in which off-diagonal residual stress components of the momentum flux are expected to be most active coincide with regions of strong mean profile gradients, where the turbulence is likely to possess a significant electromagnetic component.[15] In particular, the relevant parameter describing the lowest order correction due to finite-β effects for drift wave turbulence can be easily shown to be $\beta(qR/L_n)^2$, where $\beta \equiv c_s^2/v_A^2$, q is the safety factor, R is the major radius, and L_n is the density scale length. Thus, for regions of steep equilibrium profiles, the transport induced by drift waves is likely to be strongly modified by electromagnetic contributions. Also, we note that Ref. 2 has identified $M_A \equiv v_\phi/v_A$ and $\beta_N \equiv \beta_T a B_T/I_p$ as appropriate dimensionless quantities for describing the rate of intrinsic rotation (here β_T is the plasma pressure divided by the magnetic pressure induced by the toroidal field B_T, a is the minor radius, and I_p is the plasma current). While accurate comparisons between theory and experiment require comprehensive transport models, making any confident speculation as to the origin of this scaling trend premature at this point, the presence of magnetohydrodynamic (MHD) time scales in the scaling

of toroidal rotation provides an experimental impetus for the generalization of momentum transport formulations to the electromagnetic regime.

Aside from modifying the momentum transport induced by drift wave microturbulence, the generalization to an electromagnetic gyrokinetic framework allows for the inclusion of additional avenues of transport with a distinctly different character in comparison to the already exhaustively analyzed electrostatic limit of drift wave turbulence. In particular, kinetic Alfvén waves (KSAWs), which to lowest order propagate along magnetic field lines, provide an alternate channel, which is particularly well suited to the transport of parallel momentum. These modes may be driven directly by energetic particles, but are most likely destabilized via mode conversion of toroidal Alfvén modes (TAE) near rational surfaces.[16,17] Unlike ideal Alfvén waves, KSAWs possess both a finite radial group velocity, such that they are capable of propagating across magnetic field lines, as well as a finite component of parallel electric field. The latter property is shown below to play a critical role in determining the strength of the momentum flux. Similarly, the presence of electromagnetic turbulence introduces tantalizing new wrinkles into the momentum budget in comparison to the electrostatic case. Most notably, momentum may be deposited into the electromagnetic field, thus allowing for an additional degree of freedom when describing the evolution of parallel momentum.

In this paper, we compute the turbulent transport of parallel momentum induced by a subset of modes present within the gyrokinetic framework. Our emphasis throughout this analysis is on describing the character of off-diagonal transport in regions of steep pressure gradients, with a particular focus on the residual stress term. Specifically, as alluded to above, even for modes which are often well described in the electrostatic approximation, the steepening of the equilibrium pressure gradient in the vicinity of a transport barrier can introduce a robust electromagnetic component to the underlying microturbulence. A quasilinear calculation within a simplified geometry demonstrates that the resonant component of the momentum flux is proportional to the magnitude of the fluctuating parallel electric field. Thus, for modes which are to lowest order electrostatic, the addition of an inductive component to the parallel electric field provides a robust means of altering the level of momentum transport induced by these modes for a given spectrum $|\delta\phi_{k,\omega}|^2$ assuming a linear relation between $\delta\phi_{k,\omega}$ and $\delta A_{\parallel k,\omega}$. Similarly, for modes which are fundamentally MHD in nature, this finite $\delta E_\parallel$ requirement, limits the range of modes capable of contributing to the resonant component of the momentum flux to those which possess a nonzero value of $\delta E_\parallel$, such as, KSAWs. While the finite $\delta E_\parallel$ restriction is formally relaxed for the nonresonant component of the momentum flux, a quasilinear derivation demonstrates that the nonresonant momentum flux has a qualitatively similar structure, although different in detail.

The remainder of this paper is organized as follows: In Sec. II, a covariant formulation for the gyrokinetic Poisson–Ampere system is developed. Section III presents a systematic derivation of the parallel momentum theorem for elec-

tromagnetic gyrokinetic modes. In Sec. IV, the general structure of the radial flux of parallel momentum is derived. Section V introduces a simple reduced model for drift-Alfvén turbulence in the fluid limit. Simple asymptotic forms for the momentum flux are then derived, with more general parameter regimes treated numerically. Finally, in Sec. VI we conclude.

II. GYROKINETIC POISSON–AMPERE SYSTEM

Before proceeding further it is useful to consider some technical elements of the gyrokinetic formulation of parallel momentum transport. The radial flux of parallel momentum (considering only electrostatic fluctuations for simplicity) is given by $m_i\langle\delta(nv_\parallel)\delta v_x\rangle$, where x is a radial variable. To lowest order, this flux can be written in the form, $m_i\langle\delta(nv_\parallel)\delta v_x\rangle \approx m_i n_0\langle\delta v_\parallel\delta v_x\rangle + m_i\langle v_\parallel\rangle\langle\delta n\delta v_x\rangle$. Thus, one can see that turbulence may transport parallel momentum either via parallel Reynolds stresses, or through particle fluxes. In this paper we will focus primarily on the former of these two transport mechanisms, and leave the subtle issue of particle transport contributions to the momentum flux to a future paper. Disentangling these two transport mechanisms, is however, nontrivial since they are conflated within the gyrokinetic formulation of parallel momentum. One potential means of extricating the former of these mechanisms is to consider the evolution of parallel velocity rather than momentum. This approach is not ideal, since parallel velocity is not a conserved quantity, and thus a source term would be introduced. A much simpler approach is to boost to a frame where the plasma is locally at rest. In this way, particle fluxes will transport negligible amounts of momentum.

In order to apply this procedure, it is necessary to introduce a well defined set of parallel transformations which leave the gyrokinetic Poisson–Ampere system invariant. In the laboratory frame, for a system with a mean velocity given by $\langle v_\parallel\rangle$, the gyrokinetic Poisson–Ampere system can be approximated as (see Appendix A)

$$k_\perp^2\,\delta\phi_{k,\omega} = 4\pi\sum_s q_s \int d^3\bar{v}\left\{ J_0(\lambda)\delta F_{k,\omega}^{(s)} \right.$$
$$\left. + [J_0^2(\lambda)-1]\frac{q_s}{T_s}\langle F^{(s)}\rangle\left(\delta\phi_{k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta A_{\parallel k,\omega} \right) \right\}, \quad (1a)$$

$$k_\perp^2\,\delta A_{\parallel k,\omega} = \frac{4\pi}{c}\sum_s q_s \int d^3\bar{v}\,v_\parallel\left\{ J_0(\lambda)\delta F_{k,\omega}^{(s)} \right.$$
$$\left. + [J_0^2(\lambda)-1]\frac{q_s}{T_s}\langle F^{(s)}\rangle\left(\delta\phi_{k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta A_{\parallel k,\omega} \right) \right\}, \quad (1b)$$

where $\langle F^{(s)}\rangle$ is given by a shifted Maxwellian, $\lambda\equiv k_\perp\rho_\perp$, $\int d^3\bar{v}\equiv 2\pi\int d\mu dv_\parallel B$, and $\mu\equiv(1/2)v_\perp^2/B$. Equations (1a) and (1b) can be written in a more suggestive form by multiplying Eq. (1a) by $\langle v_\parallel\rangle/c$, then subtracting the result from Eq. (1b), yielding

$$k_\perp^2\left(\delta A_{\parallel k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta\phi_{k,\omega} \right)$$
$$= \frac{4\pi}{c}\sum_s q_s \int d^3\bar{v}(v_\parallel - \langle v_\parallel\rangle)J_0(\lambda)\delta F_{k,\omega}^{(s)}. \quad (2)$$

If we introduce the definitions $\delta\rho_{k,\omega}\equiv\Sigma_s q_s\int d\bar{v}J_0(\lambda)\delta F_{k,\omega}^{(s)}$ and $\delta j_{k,\omega}\equiv\Sigma_s q_s\int d\bar{v}v_\parallel J_0(\lambda)\delta F_{k,\omega}^{(s)}$, Eq. (2) can be rewritten as

$$k_\perp^2\left(\delta A_{\parallel k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta\phi_{k,\omega} \right) = 4\pi\left(\frac{\delta j_{k,\omega}}{c} - \frac{\langle v_\parallel\rangle}{c}\delta\rho_{k,\omega} \right). \quad (3)$$

Following a similar procedure, Poisson's equation [Eq. (1a)] can be rewritten to linear order in $\langle v_\parallel\rangle/c$, as

$$\epsilon_\perp(k)k_\perp^2\left(\delta\phi_{k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta A_{\parallel k,\omega} \right) = 4\pi\left(\delta\rho_{k,\omega} - \frac{\langle v_\parallel\rangle}{c}\frac{\delta j_{k,\omega}}{c} \right), \quad (4)$$

where $\epsilon_\perp(k)\equiv 1+\Sigma_s(k_{Ds}^2/k_\perp^2)[1-I_0(b_s)\exp(-b_s)]$ and $k_{Ds}\equiv 4\pi n_s q_s^2/T_s$. Equations (3) and (4) can be seen to have a particularly appealing form by defining the two vectors

$$\psi^\alpha\equiv\begin{pmatrix}\delta\phi_{k,\omega}\\ \delta A_{\parallel k,\omega}\end{pmatrix}, \quad j^\alpha\equiv\begin{pmatrix}c\,\delta\rho_{k,\omega}\\ \delta j_{k,\omega}\end{pmatrix},$$

which may be transformed into covariant vectors in the usual way, i.e.,

$$\psi_\alpha = g_{\alpha\beta}\psi^\beta,$$

where

$$g_{\alpha\beta}\equiv\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

To first order in $\langle v_\parallel\rangle/c$, the transformation of the two vectors may be written as

$$\bar{x}^0 = x^0 - \frac{\langle v_\parallel\rangle}{c}x^1, \quad \bar{x}^1 = x^1 - \frac{\langle v_\parallel\rangle}{c}x^0,$$

hence Eqs. (3) and (4) can be rewritten in the plasma frame as

$$k_\perp^2\,\delta\bar{A}_{\parallel k,\omega} = 4\pi\frac{\delta\bar{j}_{k,\omega}}{c}, \quad (5a)$$

$$\epsilon_\perp(k)k_\perp^2\,\delta\bar{\phi}_{k,\omega} = 4\pi\delta\bar{\rho}_{k,\omega}. \quad (5b)$$

Thus, an explicitly covariant form of the gyrokinetic Poisson–Ampere system may be introduced (see Ref. 18 for the $p_\parallel$ counterpart)

$$M_\beta^\alpha\psi^\beta = \frac{4\pi}{c}j^\alpha, \quad (6)$$

where

$$M_\beta^\alpha\equiv\begin{bmatrix}\epsilon_\perp(k)k_\perp^2 & 0\\ 0 & k_\perp^2\end{bmatrix}.$$

Note that in the drift kinetic limit where $\epsilon_\perp(k)\to 1$, Eqs. (3) and (4) can be seen to decouple into $k_\perp^2\,\delta\phi_{k,\omega}=4\pi\delta\rho_{k,\omega}$ and $k_\perp^2\,\delta A_\parallel k, \; \omega=(4\pi/c)\delta j_{k,\omega}$. Similarly, in the electrostatic

gyrokinetic limit, Eqs. (3) and (4) reduce to $\epsilon_\perp(k)k_\perp^2 \delta\phi_{k,\omega} = 4\pi\delta\rho_{k,\omega}$. Thus, while a covariant formulation is still well defined for these two cases, it is clearly not necessary. However, for the regime in which we will be interested in [electromagnetic with $b=k_\perp^2 \rho_i^2 \lesssim 1$, such that $\epsilon_\perp(k) \gg 1$], in order to maintain a set of equations invariant under parallel translations, a covariant formulation is required.

III. MOMENTUM THEOREM

Throughout this analysis we will find it useful to utilize a $v_\parallel$ gyrokinetic representation. This formulation is convenient since we are concerned with describing the velocity profile, which is not necessarily coincident with the profile of canonical momentum. Considering the electromagnetic gyrokinetic equation in the plasma frame (see Refs. 19 and 20 or Ref. 21 for a $p_\parallel$ formulation)

$$0 = \frac{\partial F_s}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\dot{\mathbf{X}} F_s) + \frac{\partial}{\partial v_\parallel}(\dot{V}_\parallel F_s),$$ (7a)

where

$$\dot{\mathbf{X}} = v_\parallel \left(\hat{b} + \frac{\langle \delta\mathbf{B}_\perp \rangle_\alpha}{B}\right) + \frac{c}{B}\hat{b} \times \nabla\langle\delta\phi\rangle_\alpha,$$ (7b)

$$\dot{V}_\parallel = \frac{-q_s}{m_s c}\frac{\partial\langle\delta A_\parallel\rangle_\alpha}{\partial t} - \frac{q_s}{m_s}\left(\hat{b} + \frac{\langle\delta\mathbf{B}_\perp\rangle_\alpha}{B}\right) \cdot \nabla\langle\delta\phi\rangle_\alpha.$$ (7c)

Here, the subscript s represents the species of particle, $\langle\cdots\rangle_\alpha \equiv (2\pi)^{-1}\int_0^{2\pi}d\alpha(\cdots)$, and for simplicity we consider cylindrical geometry. Also, note that $\delta\mathbf{A}_\perp$ contributions have been neglected, and thus our description will be incapable of describing the parallel compression of the magnetic field. This effect may be easily incorporated, but would substantially increase the algebraic complexity of the derivation.

In order to derive a general expression for the evolution of parallel momentum it is useful to separate the temporal and perpendicular spatial scales into a set of "fast" variables associated with the rapidly varying microturbulence, which we will denote by $(\mathbf{x}_\perp, t)$, and a set of "slow" variables denoted by $(\mathbf{X}_\perp, T)$, where these two sets of variables should be regarded as independent.[22] This separation allows for perpendicular space and time derivatives to be decomposed as

$$\nabla_\perp \rightarrow \nabla_\perp^{(0)} + \varepsilon\nabla_\perp^{(1)}, \quad \frac{\partial}{\partial t} \rightarrow \varepsilon\frac{\partial}{\partial t} + \varepsilon^2\frac{\partial}{\partial T},$$ (8)

with the parallel derivative ordered as

$$\hat{b}\cdot\nabla \sim \varepsilon\hat{b}\cdot\nabla,$$ (9)

where $\varepsilon \sim \rho_i/L_n$, ω_k/ω_{ci}, $k_\parallel/k_\perp$. Note that since functions of only the large scales are assumed to be uniform along magnetic field lines, there is no need to introduce an analogous decomposition in the parallel direction. While for the simple geometry utilized here, an analogous statement can be made for the poloidal variable as well, it is convenient to introduce this scale separation such that the expressions derived below have a greater range of applicability and a more transparent form. If we now assume the fluctuating fields to be approxi-

mately described by their mixing length levels, we can introduce the ordering,

$$\delta\phi = \varepsilon\delta\phi^{(1)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \varepsilon^2\delta\phi^{(2)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \cdots,$$

$$\delta A_\parallel = \varepsilon\delta A_\parallel^{(1)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \varepsilon^2\delta A_\parallel^{(2)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \cdots,$$

$$F_s = F_s^{(0)}(\mathbf{X}_\perp) + \varepsilon\delta F_s^{(1)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \varepsilon^2\delta F_s^{(2)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \cdots,$$

where for generality we allow magnetic and electrostatic fluctuations to be of the same order, and for convenience we will take $F_s^{(0)}$ to be a (centered) Maxwellian. Furthermore, we may define an average over the fast space and time scales such that $\langle\delta\psi(\mathbf{x},t,\mathbf{X}_\perp,T)\rangle=0$, but functions of only slow variables are left unaltered, i.e., $\langle\psi(\mathbf{X}_\perp,T)\rangle=\psi(\mathbf{X}_\perp,T)$. Similarly, averages over the fast scales annihilate derivatives of fast variables, as well as derivatives along magnetic field lines since we are assuming statistical homogeneity in the parallel direction, but commute with slow derivatives, i.e., $\langle\nabla_\perp^{(0)}\psi\rangle=\langle\hat{b}\cdot\nabla\psi\rangle=0$, but $\langle\nabla_\perp^{(1)}\psi\rangle=\nabla_\perp^{(1)}\langle\psi\rangle$.

Here, it is useful to derive properties of the $J_0(\lambda)$ operator, since this operator will appear frequently in the ensuing analysis. Writing this operator in terms of the fast and slow variables introduced above, we find

$$J_0(\lambda) \approx 1 + \frac{1}{4}\rho_\perp^2\{[\nabla_\perp^{(0)}]^2 + 2\varepsilon\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)} + \varepsilon^2[\nabla_\perp^{(1)}]^2\} + \cdots,$$

such that we may define

$$J_0^{(0)}(\lambda) \equiv 1 + (1/4)\rho_\perp^2[\nabla_\perp^{(0)}]^2 + \cdots,$$

$$J_0^{(1)}(\lambda) \equiv \frac{1}{2}\rho_\perp^2\nabla_\perp^{(1)}\cdot\nabla_\perp^{(0)} + \cdots.$$

Hence, while $J_0^{(0)}$ commutes with fluctuation quantities inside averages (i.e., it involves an even number of integrations by parts and the surface terms vanish), $J_0^{(1)}$ cannot be commuted without the introduction of surface terms.

A general expression for the evolution of parallel momentum can be obtained by operating on Eq. (7a) with $\Sigma_s m_s \int d^3\bar{v}v_\parallel$, and averaging over the fast scales, yielding

$$\left\langle\frac{\partial P_\parallel}{\partial t}\right\rangle + \left\langle\nabla\cdot\sum_s m_s\int d^3\bar{v}\dot{\mathbf{X}}v_\parallel F_s\right\rangle$$
$$= \left\langle\sum_s m_s\int d^3\bar{v}\dot{V}_\parallel F_s\right\rangle,$$ (10)

where $P_\parallel \equiv \Sigma_s m_s\int d^3\bar{v}v_\parallel F_s$. Equation (10) may be simplified via an expansion in ε. Considering the first term on the LHS of Eq. (10), this term can be simplified as

$$\left\langle \frac{\partial P_\parallel}{\partial t} \right\rangle = \left\langle \left(\varepsilon \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T} \right) \sum_s m_s \int d^3\bar{v} v_\parallel [F_s^{(0)}(\mathbf{X}_\perp) + \varepsilon \delta F_s^{(1)}(\mathbf{x},t,\mathbf{X}_\perp,T) + \cdots] \right\rangle$$

$$\approx \varepsilon^3 \frac{\partial}{\partial T} \left\langle \sum_s m_s \int d^3\bar{v} v_\parallel \delta F_s^{(1)}(\mathbf{x},t,\mathbf{X}_\perp,T) \right\rangle = \varepsilon^3 \frac{\partial}{\partial T} \langle \delta P_\parallel^{(1)} \rangle, \tag{11}$$

where formally $\delta F_s^{(1)}$ vanishes upon averaging, however the momentum theorem derived below will be more transparent with the inclusion of this term. Similarly, the second term on the LHS of Eq. (10) may be written to lowest order as

$$\left\langle \nabla \cdot \sum_s m_s \int d^3\bar{v} v_\parallel \dot{\mathbf{X}} F_s \right\rangle \approx \varepsilon^3 \frac{c}{B} \nabla_\perp^{(1)} \cdot \left\langle \sum_s m_s \int d^3\bar{v} v_\parallel \delta F_s^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta\phi^{(1)}] \right\rangle$$

$$- \varepsilon^3 \frac{1}{B} \nabla_\perp^{(1)} \cdot \left\langle \sum_s m_s \int d^3\bar{v} v_\parallel^2 \delta F_s^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \right\rangle, \tag{12}$$

such that the lowest order surviving term again enters at $O(\varepsilon^3)$. The first term in Eq. (12) can be recognized as describing momentum transported by $\mathbf{E} \times \mathbf{B}$ convection, and the second term arises due to magnetic flutter.

Turning now to the RHS of Eq. (10), before explicitly evaluating this term it is convenient to consider some qualitative characteristics of this expression. Considering for simplicity the electrostatic limit, the RHS of Eq. (10) can be written as

$$f_\parallel = -\left\langle \sum_s q_s \int d^3\bar{v} \delta F_s J_0(\lambda) \hat{b} \cdot \nabla \delta\phi \right\rangle. \tag{13}$$

This expression may be evaluated using the gyrokinetic Poisson equation, which may be approximated as

$$(1 + k_{Di}^2 \rho_i^2) \nabla_\perp^2 \delta\phi \approx -4\pi \sum_s q_s \int d^3\bar{v} J_0(\lambda) \delta F_s. \tag{14}$$

The first term in parentheses on the LHS of Eq. (14) appears due to deviations from quasineutrality, and vanishes in the limit of negligible Debye length. The second term, however, is nonvanishing in the limit $k_{Di} \to \infty$. This violation of gyrocenter quasineutrality results from the polarization drift appearing as an effective shielding in the gyrokinetic Poisson equation, rather than in the gyrocenter equations of motion [i.e., Eq. (7b)]. Thus, while in the drift kinetic limit [i.e., $(k_\perp \rho_i)^2 \to 0$], the RHS of Eq. (10) vanishes identically from quasineutrality, for the more general limit of $(k_\perp \rho_i)^2 \lesssim 1$, the RHS of Eq. (10) is in general nonvanishing due to the violation of gyrocenter quasineutrality. Hence, a comprehensive description of parallel momentum transport requires evaluation of this term.

To second order, the RHS of Eq. (10) can be written as

$$f_\parallel^{(2)} = \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_\parallel^{(2)} F_s^{(0)} \right\rangle, \tag{15}$$

where

$$\dot{V}_\parallel^{(2)} = -\frac{q_s}{m_s} J_0^{(0)}(\lambda) \left[\hat{b} \cdot \nabla \delta\phi^{(1)} + \frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel^{(1)} \right]$$

$$+ \frac{q_s}{m_s} \frac{1}{B} \nabla_\perp^{(0)} \cdot \{ J_0^{(0)}(\lambda) \delta\phi^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \}. \tag{16}$$

Substituting Eq. (16) into Eq. (15) yields

$$f_\parallel^{(2)} = -\left\langle \sum_s q_s \int d^3\bar{v} F_s^{(0)} J_0^{(0)}(\lambda) \left[\hat{b} \cdot \nabla \delta\phi^{(1)} + \frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel^{(1)} \right] \right\rangle$$

$$+ \frac{1}{B} \sum_s q_s \int d^3\bar{v} F_s^{(0)}$$

$$\times \langle \nabla_\perp^{(0)} \cdot \{ J_0^{(0)}(\lambda) \delta\phi^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \} \rangle = 0,$$

such that consistent with the LHS, the parallel force vanishes at second order. The third order parallel force may be written as

$$f_\parallel^{(3)} = \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_\parallel^{(2)} \delta F_s^{(1)} \right\rangle$$

$$+ \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_\parallel^{(3)} F_s^{(0)} \right\rangle, \tag{17}$$

where

$$\dot{V}_\parallel^{(3)} = -\frac{q_s}{m_s} J_0^{(0)}(\lambda) \left[\hat{b} \cdot \nabla \delta\phi^{(2)} + \frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel^{(2)} + \frac{1}{c} \frac{\partial}{\partial T} \delta A_\parallel^{(1)} \right]$$

$$- \frac{q_s}{m_s} J_0^{(1)}(\lambda) \left[\hat{b} \cdot \nabla \delta\phi^{(1)} + \frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel^{(1)} \right]$$

$$+ \frac{q_s}{m_s} \frac{1}{B} \nabla_\perp^{(1)} \cdot \{ J_0^{(0)}(\lambda) \delta\phi^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \}, \tag{18}$$

and the gyrokinetic Poisson–Ampere system is given to first order by

$$\epsilon_\perp [\nabla_\perp^{(0)}]^2 \delta\phi^{(1)} = -4\pi \sum_s q_s \int d^3\bar{v} J_0^{(0)}(\lambda) \delta F_s^{(1)}, \qquad (19a)$$

$$[\nabla_\perp^{(0)}]^2 \delta A_\parallel^{(1)} = -\frac{4\pi}{c} \sum_s q_s \int d^3\bar{v} v_\parallel J_0^{(0)}(\lambda) \delta F_s^{(1)}, \qquad (19b)$$

where $\epsilon_\perp \approx k_{Di}^2 \rho_i^2$. Considering the second term in Eq. (17) first, this term can be rewritten after substitution of Eq. (18) as

$$\left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_\parallel^{(3)} F_s^{(0)} \right\rangle$$

$$= \frac{1}{B} \sum_s q_s \int d^3\bar{v} F_s^{(0)} \nabla_\perp^{(1)}$$

$$\cdot \langle \{ J_0^{(0)}(\lambda) \delta\phi^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \} \rangle. \qquad (20)$$

Turning now to the first term in Eq. (17), Eq. (16) may be utilized, such that this term can be written in the form,

$$\left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_\parallel^{(2)} \delta F_s^{(1)} \right\rangle = -\left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} J_0^{(0)}(\lambda) \left[\hat{b} \cdot \nabla \delta\phi^{(1)} + \frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel^{(1)} \right] \right\rangle$$

$$+ \frac{1}{B} \left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} \nabla_\perp^{(0)} \cdot \{ J_0^{(0)}(\lambda) \delta\phi^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \} \right\rangle. \qquad (21)$$

The first term in Eq. (21), can be simplified via substitution of the first order Poisson equation (noting that $J_0^{(0)}$ can be commuted), i.e.,

$$-\left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} J_0^{(0)}(\lambda) \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle = \frac{\epsilon_\perp}{4\pi} \langle (\nabla_\perp^{(0)})^2 \delta\phi^{(1)} (\hat{b} \cdot \nabla) \delta\phi^{(1)} \rangle$$

$$= -\frac{\epsilon_\perp}{4\pi} \langle \nabla_\perp^{(0)} \delta\phi^{(1)} \cdot (\hat{b} \cdot \nabla) \nabla_\perp^{(0)} \delta\phi^{(1)} \rangle = -\frac{\epsilon_\perp}{8\pi} \langle \hat{b} \cdot \nabla |\nabla_\perp^{(0)} \delta\phi^{(1)}|^2 \rangle = 0.$$

Thus, after an integration by parts in time, Eq. (21) can be rewritten as

$$\left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_\parallel^{(2)} \delta F_s^{(1)} \right\rangle = -\frac{\epsilon_\perp}{4\pi c} \left\langle \delta A_\parallel^{(1)} \frac{\partial}{\partial t} [\nabla_\perp^{(0)}]^2 \delta\phi^{(1)} \right\rangle + \frac{1}{B} \left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} \nabla_\perp^{(0)} \cdot \{ J_0^{(0)}(\lambda) \delta\phi^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta A_\parallel^{(1)}] \} \right\rangle. \qquad (22)$$

The first term in Eq. (22) can again be computed from Poisson's equation. Taking the time derivative of Eq. (19a), multiplying by $\delta A_\parallel^{(1)}$, and averaging, yields

$$\frac{\epsilon_\perp}{4\pi} \left\langle \delta A_\parallel^{(1)} \frac{\partial}{\partial t} [\nabla_\perp^{(0)}]^2 \delta\phi^{(1)} \right\rangle$$

$$= -\left\langle \delta A_\parallel^{(1)} \sum_s q_s \int d^3\bar{v} J_0^{(0)}(\lambda) \frac{\partial}{\partial t} \delta F_s^{(1)} \right\rangle. \qquad (23)$$

The gyrokinetic equation to second order can be written as

$$\frac{\partial}{\partial t} \delta F_s^{(1)}$$

$$= -v_\parallel \hat{b} \cdot \nabla \delta F_s^{(1)}$$

$$- \frac{c}{B} \nabla_\perp^{(1)} \cdot \left\{ F_s^{(0)} \hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \left[\delta\phi^{(1)} - \frac{v_\parallel}{c} \delta A_\parallel^{(1)} \right] \right\}$$

$$- \frac{c}{B} \nabla_\perp^{(0)} \cdot \left\{ \delta F_s^{(1)} \hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \left[\delta\phi^{(1)} - \frac{v_\parallel}{c} \delta A_\parallel^{(1)} \right] \right\}$$

$$- \frac{q_s}{m_s} J_0^{(0)}(\lambda) \delta E_\parallel^{(2)} \frac{\partial F_s^{(0)}}{\partial v_\parallel}, \qquad (24)$$

where $\delta E_\parallel \equiv -\hat{b} \cdot \nabla \delta\phi - (1/c) \partial \delta A_\parallel / \partial t$. Inserting Eq. (24) into Eq. (23), using the first order Ampere's law given by Eq. (19b), and simplifying, allows Eq. (23) to be written as

$$\frac{\epsilon_\perp}{4\pi} \left\langle \delta A_\parallel^{(1)} \frac{\partial}{\partial t} [\nabla_\perp^{(0)}]^2 \delta\phi^{(1)} \right\rangle$$

$$= \frac{c}{B} \left\langle \delta A_\parallel^{(1)} \sum_s q_s \int d^3\bar{v} J_0^{(0)}(\lambda) \nabla_\perp^{(1)} \right.$$

$$\left. \cdot \{ F_s^{(0)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta\phi^{(1)}] \} \right\rangle$$

$$+ \frac{c}{B} \left\langle \delta A_\parallel^{(1)} \sum_s q_s \int d^3\bar{v} J_0^{(0)}(\lambda) \nabla_\perp^{(0)} \right.$$

$$\left. \cdot \{ \delta F_s^{(1)} [\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta\phi^{(1)}] \} \right\rangle. \qquad (25)$$

Substitution of Eq. (25) into Eq. (22), and integrating by parts, allows Eq. (22) to be simplified,

$$\left\langle \sum_s m_s \int d^3\bar{v}\, \dot{V}_\parallel^{(2)}\, \delta F_s^{(1)} \right\rangle$$

$$= -\frac{1}{B}\left\langle \delta A_\parallel^{(1)} \sum_s q_s \int d^3\bar{v}\, J_0^{(0)}(\lambda)\nabla_\perp^{(1)} \right.$$

$$\left. \cdot \{F_s^{(0)}[\hat{b}\times\nabla_\perp^{(0)}J_0^{(0)}(\lambda)\delta\phi^{(1)}]\} \right\rangle. \tag{26}$$

After summing Eq. (26) with Eq. (20), the parallel force $f_\parallel^{(3)}$ can be written as

$$f_\parallel^{(3)} = -\frac{1}{B}\nabla_\perp^{(1)}\cdot\left\langle \sum_s q_s \int d^3\bar{v}\, F_s^{(0)}J_0^{(0)}(\lambda)\delta A_\parallel^{(1)} \right.$$

$$\left. \times[\hat{b}\times\nabla_\perp^{(0)}J_0^{(0)}(\lambda)\delta\phi^{(1)}] \right\rangle. \tag{27}$$

Thus, to lowest nontrivial order, the evolution of parallel momentum can be described by

$$\frac{\partial\langle\delta P_\parallel\rangle}{\partial T} + \nabla_\perp^{(1)}\cdot\Pi_\parallel^{(2)} = f_\parallel^{(3)}, \tag{28}$$

where $\nabla_\perp^{(1)}\cdot\Pi_\parallel^{(2)}$ is defined by Eq. (12), and $f_\parallel^{(3)}$ by Eq. (27). The RHS of Eq. (28), which vanishes in the drift kinetic limit, can be shown to be equivalent to the parallel component of $\delta\mathbf{v}_{EB}\times\delta\mathbf{B}_\perp$, and is thus the remnant of the $\rho\mathbf{v}\times\mathbf{B}$ parallel force (note that $\hat{b}$ is in the direction of the equilibrium magnetic field). This term, while subdominant for drift wave turbulence in the limit $(k_\perp\rho_i)^2 < 1$, will be shown to play an important role in determining the radial flux of momentum induced by KSAWs, where lowest order contributions emerge due to finite Larmor radius corrections.

IV. GENERAL EXPRESSION FOR THE MOMENTUM FLUX

Here it is useful to derive a general expression for the radial flux of parallel momentum. For simplicity we consider only the third order momentum theorem given by Eq. (28), and leave higher order contributions for future analysis. Substituting the plasma response given by Eq. (B5) into Eq. (28) yields an expression for the radial component of the momentum flux,

$$\Pi_\parallel^{\text{tot}} = -i\frac{c^2}{B^2}\sum_{s,k} m_s \int d^3\bar{v}\, v_\parallel \frac{J_0^2(\lambda)}{\omega_k - k_\parallel v_\parallel} k_y^2\left(\frac{\partial}{\partial x} - \frac{\partial\langle v_\parallel\rangle}{\partial x}\frac{\partial}{\partial v_\parallel}\right)$$

$$\times F_s^{(0)}\left|\delta\phi_k - \frac{v_\parallel}{c}\delta A_{\parallel k}\right|^2 - i\frac{c}{B}\sum_{s,k}\int d^3\bar{v}\, v_\parallel \frac{J_0^2(\lambda)}{\omega_k - k_\parallel v_\parallel}$$

$$\times k_\parallel k_y \frac{\partial F_s^{(0)}}{\partial v_\parallel}\left(\delta\phi_{-k} - \frac{v_\parallel}{c}\delta A_{\parallel-k}\right)\left(\delta\phi_k - \frac{\omega_k}{ck_\parallel}\delta A_{\parallel k}\right)$$

$$+ \frac{i}{B}\sum_{s,k}\int d^3\bar{v}\, J_0^2(\lambda)F_s^{(0)}k_y\delta A_{\parallel k}\delta\phi_{-k}. \tag{29}$$

Before proceeding further, it is useful to comment on the general form of this expression. First, near wave-particle resonances where $v_\parallel \approx \omega_k/k_\parallel$, it is easy to see that both the first and second terms in Eq. (29) are explicitly proportional

to the magnitude squared of the parallel electric field (note that the third term does not contain a wave-particle resonance). This observation allows us to immediately draw qualitative conclusions on the transport induced by various electromagnetic modes. First, for ideal Alfvén modes with a dispersion relation given by $\omega_k^2 = v_A^2 k_\parallel^2$ and $\delta E_\parallel = 0$, the resonant component of Eq. (29) is seen to vanish identically. Thus, it is evident that dispersive corrections, which introduce a finite value of $\delta E_\parallel$, play a crucial role in determining the level of momentum transport induced by Alfvén waves. Similarly, for modes which are to lowest order electrostatic, such as drift wave microturbulence, finite-β corrections can either increase or reduce the level of parallel momentum transport carried by resonant particles depending on whether they enhance or compete with the electrostatic component of the parallel electric field. As a caveat, we note that the presence of a drift term within the quasilinear response function provides a potential means of weakening the link between resonant particle transport and the parallel electric field, and will be investigated in a future publication. For the case of nonresonant particles, while one may anticipate a similar trend as that discussed above, for the usual case of $v_{thi} < \omega_k/k_\parallel$, it is likely that finite-β corrections will have a less prominent effect in comparison to the resonant component.

In order to further simplify Eq. (29), the second term can be rewritten by adding and subtracting ω_k to the numerator, yielding

$$\Pi_\parallel^{\text{tot}} = -i\frac{c^2}{B^2}\sum_{s,k} m_s \int d^3\bar{v}\, v_\parallel \frac{J_0^2(\lambda)}{\omega_k - k_\parallel v_\parallel} k_y^2\left(\frac{\partial}{\partial x} - \frac{\partial\langle v_\parallel\rangle}{\partial x}\frac{\partial}{\partial v_\parallel}\right)$$

$$\times F_s^{(0)}\left|\delta\phi_k - \frac{v_\parallel}{c}\delta A_{\parallel k}\right|^2 - ie\frac{c}{B}\sum_{s,k}\int d^3\bar{v}\, v_\parallel$$

$$\times \frac{J_0^2(\lambda)}{\omega_k - k_\parallel v_\parallel} k_\parallel k_y \frac{\partial F_s^{(0)}}{\partial v_\parallel}\left[|\delta\phi_k|^2 + \left(\frac{\omega_k}{ck_\parallel}\right)\frac{v_\parallel}{c}|\delta A_{\parallel k}|^2 - \left(\frac{\omega_k}{ck_\parallel}\right)\right.$$

$$\left.\times(\delta\phi_{-k}\delta A_{\parallel k} + \delta\phi_k\delta A_{\parallel-k})\right] - i\frac{1}{B}\sum_{s,k} q_s \int d^3\bar{v}\, J_0^2(\lambda)k_y F_s^{(0)}$$

$$\times(\delta\phi_k\delta A_{\parallel-k} - \delta A_{\parallel k}\delta\phi_{-k}), \tag{30}$$

where both the real and imaginary components of the cross terms of $\delta\phi_k$ and $\delta A_{\parallel k}$ can be seen to enter into the expression for the momentum flux. From Eq. (30) the total momentum flux can be rewritten in terms of the plasma dispersion function as

$$\Pi_\parallel^{\text{tot}} = \Pi_\parallel^{\text{ES}} + \Pi_\parallel^{\text{Re}(A\phi)} + \Pi_\parallel^{\text{Im}(A\phi)} + \Pi_\parallel^{AA}, \tag{31a}$$

where

$$\Pi_\parallel^{\text{ES}} = in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}(k_y\rho_s)^2 \frac{k_\parallel}{k_y}\hat{Z}(\zeta)\Gamma_0(b)\left|\frac{e\,\delta\phi_k}{T_e}\right|^2, \tag{31b}$$

$$\Pi_\parallel^{\text{Re}(A\phi)} = -2in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\left(\frac{\omega_k}{ck_\parallel}\right)\hat{Z}(\zeta)\Gamma_0(b)\frac{e^2}{T_e^2}$$

$$\times \text{Re}(\delta A_{\parallel k}\delta\phi_{-k}), \tag{31c}$$

$$\Pi_{\parallel}^{\text{Im}(A\phi)} = 2n_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k} \frac{k_{\parallel}}{k_y} (k_y\rho_s)^2 \left(\frac{\omega_k}{ck_{\parallel}}\right)[1-\Gamma_0(b)]$$

$$\times \frac{e^2}{T_e^2} \text{Im}(\delta A_{\parallel k}\delta\phi_{-k}), \tag{31d}$$

$$\Pi_{\parallel}^{AA} = in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k} \frac{k_{\parallel}}{k_y} (k_y\rho_s)^2 \left(\frac{\omega_k}{ck_{\parallel}}\right)^2 \Gamma_0(b) \left|\frac{e\,\delta A_{\parallel k}}{T_e}\right|^2$$

$$\times \left\{ \hat{Z}(\zeta) - \left[1 + \tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c}\right)\right.\right.$$

$$\left.\left. - \frac{k_y}{k_{\parallel}}\frac{1}{\omega_{ci}}\frac{\partial\langle v_{\parallel}\rangle}{\partial x}\right] + \frac{1}{\Gamma_0(b)}\left[1 - \frac{\omega_e^*}{\omega_k}(1+\eta_e)\right]\right\}, \tag{31e}$$

$\hat{Z}(\zeta)$ is defined as

$$\hat{Z}(\zeta) \equiv -2\zeta^2\left\{\left[1 + \tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 - \frac{\eta_i}{\eta_c} + \eta_i\zeta^2\right) - \frac{k_y}{k_{\parallel}}\frac{1}{\omega_{ci}}\frac{\partial\langle v_{\parallel}\rangle}{\partial x}\right]\right.$$

$$\left. \times[1+\zeta Z(\zeta)] + \frac{1}{2}\tau^{-1}\eta_i\frac{\omega_e^*}{\omega_k}\right\}, \tag{32}$$

and it is understood that only the real part of the momentum flux is kept. Here $Z(\zeta)$ is the plasma dispersion function, $\zeta \equiv (1/\sqrt{2})(\omega_k/v_{thi}k_{\parallel})$, $\eta_c^{-1} \equiv 1/2 + b[1 - I_1(b)/I_0(b)]$, $b \equiv k_{\perp}^2\rho_i^2$, $\eta_i \equiv L_n/L_{T_i}$, $\eta_e \equiv L_n/L_{T_e}$, $L_n^{-1} \equiv -d\ln n_0/dx$, $L_{T_i}^{-1} \equiv -d\ln T_i/dx$, $L_{T_e}^{-1} \equiv -d\ln T_e/dx$, $\omega_e^* \equiv k_y c_s\rho_s/L_n$, $\omega_i^* \equiv -\tau^{-1}\omega_e^*$, $\tau \equiv T_e/T_i$, $c_s \equiv \sqrt{T_e/m_i}$, $\rho_s \equiv c_s/\omega_{ci}$, and $\omega_{ci} \equiv eB/(m_ic)$. The $\Pi_{\parallel}^{AA}$ term may be simplified significantly by noting that only the real piece contributes to the overall momentum flux. Dropping terms which are manifestly imaginary (i.e., independent of $\omega_k = \omega_k^{(r)} + i\gamma_k$), yields the simplified form,

$$\Pi_{\parallel}^{AA} = in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k} \frac{k_{\parallel}}{k_y} (k_y\rho_s)^2 \left(\frac{\omega_k}{ck_{\parallel}}\right)^2 \Gamma_0(b) \left|\frac{e\,\delta A_{\parallel k}}{T_e}\right|^2$$

$$\times \left(\hat{Z}(\zeta) - \left\{\left[\frac{\Gamma_0(b)-1}{\Gamma_0(b)}\right] - \frac{k_y}{k_{\parallel}}\frac{1}{\omega_{ci}}\frac{\partial\langle v_{\parallel}\rangle}{\partial x}\right\}\right). \tag{33}$$

Here it is useful to point out that unlike the electrostatic case in which the momentum flux is due solely to ions in the limit $m_e \to 0$, in the electromagnetic regime, electrons are capable of impacting the momentum flux, as demonstrated by Eqs. (31d) and (33) [terms independent of $\Gamma_0(b)$]. This can easily be seen to follow from magnetic flutter terms being weighted by $v_{\parallel}/c$, and thus introducing higher order moments of the electron distribution function in comparison to the electrostatic case. Since $v_{the} = \sqrt{T_e/m_e}$, as the order of the moments of the electron distribution function are increased, contributions from electrons become increasingly relevant, and as is verified below, are capable of substantially impacting momentum transport.

Equations (31a)–(31d) and (33) provide general expressions for describing the radial flux of parallel momentum induced by small scale microturbulence in the large aspect ratio limit. In order to write these expressions in a more transparent form, it is useful to separate contributions due to resonant particle scattering, and those which may be recovered in the purely fluid limit, i.e., nonresonant transport. In order to illustrate the origin of these two components more explicitly, it is useful to consider the general form of the plasma dispersion function. Symbolically, these contributions may be written using the notation

$$\text{Re}\left\{i\int dv_{\parallel}\frac{1}{\omega_k - k_{\parallel}v_{\parallel} + i\epsilon}h(v_{\parallel})\right\}, \tag{34}$$

where $h(v_{\parallel})$ is a general function of $v_{\parallel}$ and $i\epsilon$ is a small broadening term, approximated below as the growth rate of the underlying modes. For small ϵ, this integral may be approximated via separation into its resonant and nonresonant contributions, i.e.,

$$\text{Re}\left\{i\int_{v_{\parallel}\neq\omega_k/k_{\parallel}} dv_{\parallel}\frac{h(v_{\parallel})}{\omega_k - k_{\parallel}v_{\parallel} + i\epsilon}\right\} + \pi\int dv_{\parallel}\delta(\omega_k - k_{\parallel}v_{\parallel})h(v_{\parallel}). \tag{35}$$

Since the wave-particle resonance has been removed from the first term in Eq. (35), it may be approximated via the expansion

$$\text{Re}\left\{i\int_{v_{\parallel}\neq\omega_k/k_{\parallel}} dv_{\parallel}\frac{h(v_{\parallel})}{\omega_k - k_{\parallel}v_{\parallel} + i\epsilon}\right\}$$

$$\approx \frac{\epsilon}{\omega_k^2}\int dv_{\parallel}h(v_{\parallel})\left\{1 + 2\left(\frac{k_{\parallel}v_{\parallel}}{\omega_k}\right) + 3\left(\frac{k_{\parallel}v_{\parallel}}{\omega_k}\right)^2 + \cdots\right\}. \tag{36}$$

Here, the second term in Eq. (35) can be recognized as transport induced by wave-particle resonances, whereas the first term is clearly fluid in nature. In the following sections we consider both of these contributions in turn.

A. Nonresonant momentum flux

In order to compute the nonresonant component of the momentum transport it will be necessary to evaluate the principle value component of the plasma dispersion function. This can most easily be done by expanding the plasma's dispersion function in the parameter $v_{thi}k_{\parallel}/\omega_k$. Since we are primarily interested in drift waves near steep equilibrium pressure gradients, it is appropriate to consider the limit $v_{thi}k_{\parallel}/\omega_k < 1$. Expanding out the nonresonant component of $\hat{Z}(\zeta)$ in the above limit, yields

$$\hat{Z}(\zeta) \approx 1 + \tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c}\right) - \frac{k_y}{k_{\parallel}}\frac{1}{\omega_{ci}}\frac{\partial\langle v_{\parallel}\rangle}{\partial x} + O\left(\frac{v_{thi}k_{\parallel}}{\omega_k}\right)^2. \tag{37}$$

Utilizing this expression, the electrostatic component of the momentum flux given by Eq. (31b) can be expressed as

$$\Pi_\parallel^{ES} = in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}(k_y\rho_s)^2\frac{k_\parallel}{k_y}\Gamma_0(b)\left|\frac{e\,\delta\phi_k}{T_e}\right|^2$$

$$\times\left[1 + \tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c}\right) - \frac{k_y}{k_\parallel}\frac{1}{\omega_{ci}}\frac{\partial\langle v_\parallel\rangle}{\partial x}\right.$$

$$\left. + O\left(\frac{v_{thi}k_\parallel}{\omega_k}\right)^2\right], \tag{38}$$

in agreement with Ref. 7. Considering the form of Eq. (38), it is apparent that both a residual stress term as well as a diffusive term are present. As is evident from the form of Eq. (38), and noted by numerous authors previously,[7,8,23–25] the electrostatic residual stress is odd in $k_\parallel$, and hence sums to zero identically in the absence of $k_\parallel$ symmetry breaking.

The lowest order finite-β contribution to the momentum flux is given by $\Pi_\parallel^{Re(A\phi)}$, and can be approximated as

$$\Pi_\parallel^{Re(A\phi)} = -2in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\left(\frac{\omega_k}{ck_\parallel}\right)\Gamma_0(b)\frac{e^2}{T_e^2}$$

$$\times\mathrm{Re}(\delta A_{\parallel k}\delta\phi_{-k})\left[1 + \tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c}\right)\right.$$

$$\left. - \frac{k_y}{k_\parallel}\frac{1}{\omega_{ci}}\frac{\partial\langle v_\parallel\rangle}{\partial x} + O\left(\frac{v_{thi}k_\parallel}{\omega_k}\right)^2\right]. \tag{39}$$

In order to express this term in a more transparent form it will be convenient to utilize the quasilinear relation between $\delta A_{\parallel k}$ and $\delta\phi_k$, i.e.,

$$\delta A_{\parallel k} = -\left.\frac{\chi_{k,\omega}^{A\phi}}{1 + \chi_{k,\omega}^{AA}}\right|_{\omega = \omega_k}\delta\phi_k = -\frac{\chi_k^{A\phi}}{1 + \chi_k^{AA}}\delta\phi_k, \tag{40}$$

such that the transport coefficients may be written in terms of a single scalar field. Dropping terms from Eq. (39) which are pure imaginary, and assuming the turbulence is near marginality such that $|\omega_k| \gg |\gamma_k|$, yields an expression for the lowest order finite-β correction

$$\Pi_\parallel^{Re(A\phi)} = 2in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\left(\frac{\omega_k}{ck_\parallel}\right)$$

$$\times\frac{\mathrm{Re}\,\chi_k^{A\phi}}{1 + \mathrm{Re}\,\chi_k^{AA}}\Gamma_0(b)\left|\frac{e\,\delta\phi_k}{T_e}\right|^2$$

$$\times\left[\tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c}\right) + O\left(\frac{v_{thi}k_\parallel}{\omega_k}\right)^2\right]. \tag{41}$$

Considering now $\Pi_\parallel^{Im(A\phi)}$, while this term enters at the same order in β as $\Pi_\parallel^{Re(A\phi)}$ [Eq. (31d)], it is easy to see that it vanishes in the absence of finite Larmor radius corrections, and will thus typically be small in comparison to $\Pi_\parallel^{Re(A\phi)}$. An important exception corresponds to the case of Alfvén waves (discussed below), where it is easy to see that the lowest order surviving contributions to the residual stress component of the momentum flux emerge due to finite Larmor radius corrections. Rewriting Eq. (31d) in terms of $\delta\phi_k$, yields the expression

$$\Pi_\parallel^{Im(A\phi)} = 2n_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\frac{[1 - \Gamma_0(b)]}{(1 + \mathrm{Re}\,\chi_k^{AA})^2}\left|\frac{e\,\delta\phi_k}{T_e}\right|^2$$

$$\times\left[\left(\frac{\omega_k}{ck_\parallel}\right)\mathrm{Im}\,\chi_k^{A\phi} + \frac{\gamma_k}{\omega_k}(\mathrm{Re}\,\chi_k^{AA})^2\right], \tag{42}$$

where the imaginary components of the susceptibility matrix are defined in Appendix B. The remaining term given by $\Pi_\parallel^{AA}$ can be written as

$$\Pi_\parallel^{AA} = in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\left(\frac{\omega_k}{ck_\parallel}\right)^2\frac{(\mathrm{Re}\,\chi_k^{A\phi})^2}{(1 + \mathrm{Re}\,\chi_k^{AA})^2}\Gamma_0(b)$$

$$\times\left|\frac{e\,\delta\phi_k}{T_e}\right|^2\left[1 + O\left(\frac{v_{thi}k_\parallel}{\omega_k}\right)^2\right]$$

$$+ in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2[1 - \Gamma_0(b)]$$

$$\times\left(\frac{\omega_k}{ck_\parallel}\right)^2\frac{(\mathrm{Re}\,\chi_k^{A\phi})^2}{(1 + \mathrm{Re}\,\chi_k^{AA})^2}\left|\frac{e\,\delta\phi_k}{T_e}\right|^2, \tag{43}$$

which may be simplified to

$$\Pi_\parallel^{AA} = in_0 m_i c_s^2 \sum_k \frac{\omega_{ci}}{\omega_k}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\left(\frac{\omega_k}{ck_\parallel}\right)^2\frac{(\mathrm{Re}\,\chi_k^{A\phi})^2}{(1 + \mathrm{Re}\,\chi_k^{AA})^2}\left|\frac{e\,\delta\phi_k}{T_e}\right|^2, \tag{44}$$

such that the lowest order surviving piece of this term can be seen to result from the transport of electron momentum via magnetic flutter. While this term enters at higher order in $[\omega_k/(ck_\parallel)]\mathrm{Re}\,\chi_k^{A\phi}$ in comparison to Eqs. (41) and (42), it will be seen to play an important role in determining the structure of the residual stress term. After summation of Eqs. (38), (41), (42), and (44), the fluid component of the momentum flux may be approximated as

$$\Pi_\parallel = -\chi_\phi\frac{\partial\langle v_\parallel\rangle}{\partial x} + S, \tag{45}$$

where

$$\chi_\phi = n_0 m_i c_s^2 \sum_k \frac{|\gamma_k|}{\omega_k^2}(k_y\rho_s)^2\Gamma_0(b)\left|\frac{e\,\delta\phi_k}{T_e}\right|^2, \tag{46}$$

$$S = \sum_k S_k\left[A(\hat\beta) + 2B(\hat\beta)\tau^{-1}\frac{\omega_e^*}{\omega_k}\left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c}\right)\right], \tag{47}$$

$$S_k \equiv n_0 m_i c_s^2 \frac{|\gamma_k|\omega_{ci}}{\omega_k^2}\frac{k_\parallel}{k_y}(k_y\rho_s)^2\Gamma_0(b)\left|\frac{e\,\delta\phi_k}{T_e}\right|^2,$$

$$A(\hat\beta) \equiv 1 - \frac{1}{\Gamma_0(b)}\frac{1}{(1 + \mathrm{Re}\,\chi_k^{AA})^2}\left\{(\mathrm{Re}\,\chi_k^{AA})^2\right.$$

$$\left. - 2[1 - \Gamma_0(b)]\left[\frac{1}{k_\perp^2\rho_s^2}\hat\beta\left(\frac{\hat\omega_k}{\hat k_\parallel}\right)^2 + (\mathrm{Re}\,\chi_k^{AA})^2\right]\right\}, \tag{48}$$

$$B(\hat{\beta}) \equiv \frac{1}{1 + \mathrm{Re}\,\chi_k^{AA}}, \tag{49}$$

$$\mathrm{Re}\,\chi_k^{AA} \approx -\frac{1}{k_\perp^2 \rho_s^2} \hat{\beta} \left(\frac{\hat{\omega}_k}{\hat{k}_\parallel} \right)^2 \left(1 - \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \right). \tag{50}$$

Here $\mathrm{Re}\,\chi_k^{AA}$ has been rewritten in the dimensionless variables given by $\hat{\beta} \equiv \beta(qR/L_n)^2$, $\hat{k}_\parallel \equiv qRk_\parallel$, $\hat{\omega}_e^* \equiv L_n\omega_e^*/c_s$, $\hat{\omega}_{pi}^* \equiv -\tau^{-1}\hat{\omega}_e^*(1+\eta_i)$, $\hat{\omega}_k \equiv L_n\omega_k/c_s$, we have used the expression $[\omega_k^{(r)}/ck_\parallel]\mathrm{Re}\,\chi_k^{A\phi} = -\mathrm{Re}\,\chi_k^{AA}$, and taken the real piece such that ω_k is now defined as $\omega_k = \omega_k^{(r)}$.

Before performing a more detailed analysis of Eq. (45), it is useful to briefly consider some qualitative characteristics of this expression. As can be seen from Eq. (46), the diffusive term remains unaffected by finite-$\hat{\beta}$ effects to within the order kept (i.e., we have only kept terms proportional to $\hat{\beta}$ and dropped β corrections). Also, as with the electrostatic case, the electromagnetic residual stress term requires symmetry breaking in $k_\parallel$ in order to prevent Eq. (47) from vanishing after summation. More interestingly, unlike the diffusive term, the residual stress term given by Eq. (47) can potentially be significantly modified by finite-$\hat{\beta}$ effects through the coefficients $A(\hat{\beta})$ and $B(\hat{\beta})$. While the functional form of $A(\hat{\beta})$ is strongly model dependent, such that we will postpone analysis of this term, some immediate observations can be made about the behavior of $B(\hat{\beta})$. First, the functional form of $B(\hat{\beta})$ can be seen to depend sensitively on the sign of $\mathrm{Re}\,\chi_k^{AA}$. For drift wave turbulence, if we assume $\omega_k \approx \omega_e^*/(1+k_\perp^2\rho_s^2)$ for the purposes of obtaining the sign, and order of magnitude of $\mathrm{Re}\,\chi_k^{AA}$ (a more detailed analysis is presented below), we obtain the expression $\mathrm{Re}\,\chi_k^{AA} \approx \hat{\beta}(\hat{\omega}_k/\hat{k}_\parallel)^2 > 0$. Thus, $B(\hat{\beta})$ is a decreasing function of $\hat{\beta}$, which results in the magnitude of the second term in curly braces in Eq. (47) being significantly reduced for high $\hat{\beta}$. Turning now to ITG turbulence, if we assume $0 > \hat{\omega}_k \gtrsim \hat{\omega}_{pi}^*$, then $\mathrm{Re}\,\chi_k^{AA}$ is bounded by

$$\mathrm{Re}\,\chi_k^{AA} \leq -\frac{1}{k_\perp^2 \rho_s^2} \hat{\beta} \left(\frac{\hat{\omega}_k}{\hat{k}_\parallel} \right)^2 \left(\frac{1+\tau+\eta_i}{1+\eta_i} \right). \tag{51}$$

Thus, $\mathrm{Re}\,\chi_k^{AA} < 0$ and $B(\hat{\beta})$ is an increasing function of $\hat{\beta}$. Hence the magnitude of this term may be anticipated to be enhanced for ITG turbulence. This trend, is however, only valid for regimes in which $|\mathrm{Re}\,\chi_k^{AA}| < 1$, since for the form of $B(\hat{\beta})$ given by Eq. (49), $B(\hat{\beta})$ has a singularity as $\mathrm{Re}\,\chi_k^{AA} \to -1$. This singularity may be removed by noting that we have assumed $\mathrm{Re}\,\chi_k^{AA} \gg \mathrm{Im}\,\chi_k^{AA}$ (i.e., near marginality and weak wave particle interaction, see Appendix B), such that we have approximated the real part of the coefficient of Eq. (40) as $-\mathrm{Re}\,\chi_k^{A\phi}/(1+\mathrm{Re}\,\chi_k^{AA})$. Relaxing the above assumption, the denominator of the coefficient in Eq. (40) takes the form $(1+\mathrm{Re}\,\chi_k^{AA})^2 + (\mathrm{Im}\,\chi_k^{AA})^2$, such that $B(\hat{\beta})$ becomes a decreasing function of $\hat{\beta}$, regardless of the sign of $\mathrm{Re}\,\chi_k^{AA}$, for $|\mathrm{Re}\,\chi_k^{AA}| > 1$. Thus, as $\mathrm{Re}\,\chi_k^{AA}$ approaches 1, the amplitude of $B(\hat{\beta})$, and hence the dominant portion of the residual stress (since this term is typically larger than the first term for

$0 > \hat{\omega}_k \gtrsim \hat{\omega}_{pi}^*$), becomes extremely sensitive to the value of $\mathrm{Im}\,\chi_k^{AA}$, and hence the eddy decorrelation time τ_c^{-1}, here approximated by the linear growth rate $|\gamma_k|$. From this cursory analysis it can be concluded that while at least a portion of the residual stress is potentially significantly reduced due to finite-$\hat{\beta}$ terms for drift wave turbulence, for ITG turbulence, the residual stress may be modestly enhanced, whose maximum value depends sensitively on τ_c^{-1}.

B. Resonant component

Considering the resonant component of Eq. (31a), after evaluation of integrals over velocity space, one is straightforwardly led to the expression,

$$\Pi_\parallel^{\mathrm{tot}} = \sqrt{\frac{\pi}{2}} n_0 m_i c_s^2 \sum_k \frac{\Gamma_0(b)}{|k_\parallel|\rho_i} (k_y\rho_s)^2 \left(\frac{\omega_k}{k_\parallel v_{thi}} \right)$$

$$\times \exp\left[-\frac{1}{2} \left(\frac{\omega_k}{k_\parallel v_{thi}} \right)^2 \right]$$

$$\times \left[\left[\left| \frac{e\,\delta\phi_k}{T_e} \right|^2 - \left(\frac{\omega_k}{k_\parallel c} \right) \frac{e^2}{T_e^2} \delta\phi_{-k}\delta A_{\parallel k} \right. \right.$$

$$\left. - \left(\frac{\omega_k}{k_\parallel c} \right) \frac{e^2}{T_e^2} \delta\phi_k \delta A_{\parallel -k} + \left(\frac{\omega_k}{k_\parallel c} \right)^2 \left| \frac{e\,\delta A_{\parallel k}}{T_e} \right|^2 \right]$$

$$\left. \times \left[\frac{\rho_i}{L_\perp} + \frac{k_\parallel}{k_y} \left(\frac{\omega_k}{k_\parallel v_{thi}} \right) - \left(\frac{\omega_k}{k_\parallel v_{thi}} \right) \frac{1}{\omega_{ci}} \frac{\partial\langle v_\parallel\rangle}{\partial x} \right], \tag{52}$$

where

$$\frac{1}{L_\perp} \equiv \frac{1}{L_n} - \left\{ \frac{1}{2} + b\left[1 - \frac{I_1(b)}{I_0(b)} \right] - \frac{1}{2} \left(\frac{\omega_k}{k_\parallel v_{thi}} \right)^2 \right\} \frac{1}{L_{\mathrm{Ti}}}.$$

Equation (52) can be simplified in an analogous manner as the nonresonant component, however before proceeding further, it is useful to discuss a few simple limits for which this form is particularly convenient. First, for the case of ideal Alfvén waves $[\delta A_{\parallel k} = (k_\parallel c/\omega_k)\delta\phi_k]$, Eq. (52) can be seen to go to zero identically. This can easily be seen to follow due to $\delta E_\parallel$ vanishing identically in this limit. This complete cancellation, however, can be removed via the introduction of finite Larmor radius corrections, i.e., for kinetic Alfvén waves $\delta A_{\parallel k} = (k_\parallel c/\omega_k)(1+k_\perp^2\rho_s^2)\delta\phi_k$, such that the parallel electric field can be approximated as $\delta E_\parallel(k) \approx ik_\parallel k_\perp^2 \rho_s^2 \delta\phi_k$, and a finite value of momentum flux remains, albeit at a significantly reduced level.

Equation (52) may be simplified via the use of Eq. (40), and noting $[\omega_k^{(r)}/k_\parallel c]\mathrm{Re}\,\chi_k^{A\phi} = -\mathrm{Re}\,\chi_k^{AA}$, such that the resonant component may be written in the simple form,

$$\Pi_\parallel^{\mathrm{tot}} = \sum_k \frac{\Pi_{\parallel k}^{\mathrm{ES}}}{(1 + \mathrm{Re}\,\chi_k^{AA})^2}, \tag{53}$$

where the electrostatic resonant particle flux is defined by

$$\Pi_\parallel^{\mathrm{ES}} \equiv \sum_k \Pi_{\parallel k}^{\mathrm{ES}},$$

and

$$\Pi_{\parallel k}^{\mathrm{ES}} = \sqrt{\frac{\pi}{2}} n_0 m_i c_s^2 \frac{\Gamma_0(b)}{|k_\parallel| \rho_i} (k_y \rho_s)^2 \left(\frac{\omega_k}{k_\parallel v_{thi}} \right) \left| \frac{e \delta \phi_k}{T_e} \right|^2$$

$$\times \exp\left[-\frac{1}{2} \left(\frac{\omega_k}{k_\parallel v_{thi}} \right)^2 \right]$$

$$\times \left[\frac{\rho_i}{L_\perp} + \frac{k_\parallel}{k_y} \left(\frac{\omega_k}{k_\parallel v_{thi}} \right) - \left(\frac{\omega_k}{k_\parallel v_{thi}} \right) \frac{1}{\omega_{ci}} \frac{\partial \langle v_\parallel \rangle}{\partial x} \right].$$

From Eq. (53), one can see that the qualitative result for resonant particles is similar to that of the nonresonant component, except that both the residual stress and diffusive terms are strongly modified by finite-$\hat{\beta}$ corrections, and that the finite-$\hat{\beta}$ modifications have a somewhat more transparent form. Specifically, the effect of finite-$\hat{\beta}$ modifications is to introduce a coefficient proportional to $B(\hat{\beta})^2$, such that the brief analysis presented within the previous section can be carried over with little amendment. Namely, for drift wave turbulence (i.e., $0 < \omega_k < \omega_e^*$) the magnitude of Eq. (53) decreases with $\hat{\beta}$, but for ITG ($0 > \omega_k \gtrsim \omega_{pi}^*$), the magnitude is an increasing function of $\hat{\beta}$, to lowest order in $\hat{\beta}$. Note, that these trends are somewhat strengthened due to the magnitude of the finite-$\hat{\beta}$ coefficient being squared.

V. DRIFT-ALFVÉN WAVES

Expressions for the radial flux of parallel momentum were derived for general gyrokinetic modes in a simplified geometry in the previous section. While some qualitative trends were apparent, here it is useful to explicitly evaluate these terms for the case of drift-Alfvén turbulence in various limits. For simplicity, we will treat the fluid limit since this case is the most transparent. Note that within the homogeneous analysis presented below we will be unable to specify the radial eigenmode structure, and hence unable to accurately evaluate integrals over $k_\parallel$. Thus, our emphasis in the following sections will be on identifying how finite-$\hat{\beta}$ terms modify electrostatic transport coefficients. Asymptotic forms for the radial eigenmodes for both the drift wave and Alfvén wave branches are derived in Appendix C, such that the role of $\mathbf{E} \times \mathbf{B}$ flow shear as a symmetry breaking mechanism for each branch of turbulence is made evident.

As discussed in the previous section the impact of finite-$\hat{\beta}$ terms is strongly mode dependent, and often highly sensitive to the particulars of the model chosen. Here, it is instructive to consider a simple model for drift-Alfvén waves, such that explicit forms for the expressions describing finite-$\hat{\beta}$ modifications can be obtained. Since we are primarily interested in regimes for which $|v_{thi} k_\parallel / \omega_k| \ll 1$, it will be useful to truncate the expansion of the plasma dispersion function at lowest nontrivial order, such that the drift-Alfvén dispersion relationship may be written as
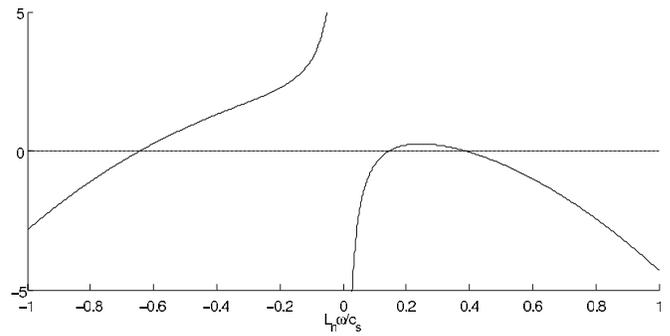
FIG. 1. Plot of roots from the drift-Alfvén dispersion relation for the parameters: $\tau=2$, $\eta_i=2$, $k_\perp \rho_s=0.3$, $\hat{\omega}_e^*=0.3/\sqrt{2}$, $\hat{k}_\parallel=1.0$, and $\hat{\beta}=5.0$.

$$0 = 1 + \tau[1 - \Gamma_0(b)] - \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \Gamma_0(b)$$

$$- \hat{\beta} \left(\frac{\hat{\omega}_k}{\hat{k}_\parallel} \right)^2 \frac{1 - \Gamma_0(b)}{k_\perp^2 \rho_s^2} \left(1 - \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \right) \left(\tau + \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \right)$$

$$+ (k_\perp \rho_i)^2 \eta_i \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \Gamma_0(b) \left[1 - \frac{1}{k_\perp^2 \rho_s^2} \hat{\beta} \left(\frac{\hat{\omega}_k}{\hat{k}_\parallel} \right)^2 \left(1 - \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \right) \right]$$

$$\times \left[1 - \frac{I_1(b)}{I_0(b)} \right]. \tag{54}$$

From Fig. 1 it is clear that Eq. (54) possesses three roots: two high frequency roots which we will refer to as Alfvén roots, as well as one drift root, which is substantially modified by finite-$\hat{\beta}$ coupling. Before proceeding further it is useful to expand Eq. (54) in the parameter $k_\perp^2 \rho_i^2 < 1$, yielding

$$0 = k_\perp^2 \rho_s^2 \left(1 - \frac{\hat{\omega}_{pi}^*}{\hat{\omega}_k} \right) + \left(1 - \frac{\hat{\omega}_e^*}{\hat{\omega}_k} \right) \left[1 - \hat{\beta} \left(\frac{\hat{\omega}_k}{\hat{k}_\parallel} \right)^2 \left(1 - \frac{\hat{\omega}_{pi}^*}{\hat{\omega}_k} \right) \right],$$

$$\tag{55}$$

which will provide a useful expression for simplified analytic estimates provided in the next subsections.

A. Drift wave branch

As discussed above, off-diagonal transport arising from the drift wave root can be substantially modified by finite-$\hat{\beta}$ effects. In order to understand the strength of finite-$\hat{\beta}$ modifications to the residual stress term, it is instructive to compute the functional forms of $A(\hat{\beta})$ and $B(\hat{\beta})$ for the simplified model given by Eq. (54). Plots of both $A(\hat{\beta})$ and $B(\hat{\beta})$ are shown in Fig. 2. As can clearly be observed, both $A(\hat{\beta})$ and $B(\hat{\beta})$ are decreasing functions of $\hat{\beta}$, with $A(\hat{\beta})$ decreasing somewhat more slowly. This strong $\hat{\beta}$ dependence suggests that during barrier formation, the level of parallel momentum transport induced by drift wave turbulence may be significantly reduced, thus providing an alternate means of quenching off-diagonal momentum transport aside from $\mathbf{E} \times \mathbf{B}$ shear decorrelation.[11]
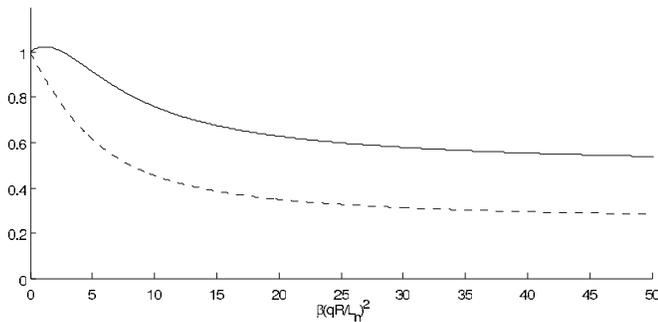
FIG. 2. Plot of quenching factor for the parameters: $\tau=1$, $\eta_i=2$, $k_\perp\rho_i=0.3$, $\hat{\omega}_e^*=0.3/\sqrt{2}$, and $\hat{k}_\parallel=1.0$. The solid and broken lines correspond to $A(\hat{\beta})$ and $B(\hat{\beta})$, respectively.

FIG. 3. Plot of the residual stress in arbitrary units vs $\hat{\beta}$ for the parameters: $\tau=1$, $k_\perp\rho_i=0.3$, $\hat{\omega}_e^*=0.3/\sqrt{2}$, and $\eta_i=2$. The solid line indicates the value of the residual stress for drift wave turbulence, the broken line corresponds to ω_k^+, and the broken dotted line indicates ω_k^-.

B. Alfvénic branches

Before considering the general form of the momentum flux induced by Alfvén waves with arbitrary $k_\perp^2\rho_s^2$, and plasmas with steep equilibrium profiles, it is useful to consider some simplified limits for which transparent analytic expressions can be derived. For these simplified limits it will be useful to utilize Eq. (55) in order to approximate Re χ_k^{AA}, yielding

$$\text{Re }\chi_k^{AA} = \hat{\beta}(\hat{\omega}_k/\hat{k}_\parallel)^2(1-\hat{\omega}_{pi}^*/\hat{\omega}_k)[1-\hat{\beta}(\hat{\omega}_k/\hat{k}_\parallel)^2(1-\hat{\omega}_{pi}^*/\hat{\omega}_k)]^{-1}. \tag{56}$$

Considering first the idealized case of $(k_\perp^2\rho_s^2, L_n^{-1}, L_{T_i}^{-1}, L_{T_e}^{-1}) \to 0$, the two Alfvén roots reduce to $\omega_k^2=k_\parallel^2 v_A^2$. Thus, in this purely Alfvénic limit $\hat{\beta}(\hat{\omega}_k/\hat{k}_\parallel)^2 \to 1$, which leads to Re $\chi_k^{AA} \to \infty$, and the residual stress term vanishes identically. Next, considering the limit of finite dispersion, but with vanishing equilibrium gradients, Eq. (55) can easily be recognized to reduce to the kinetic shear Alfvén wave dispersion relationship, i.e., $\omega_k^2=k_\parallel^2 v_A^2(1+k_\perp^2\rho_s^2)$. In this limit $(1+\text{Re }\chi_k^{AA})^{-1}=-k_\perp^2\rho_s^2$, and after a straightforward calculation, the residual stress reduces to

$$S = -2\sum_k (k_\perp\rho_s)^2\left(1-\frac{\tau^{-1}}{2}\right)S_k. \tag{57}$$

Here we do not distinguish between the positive and negative roots, since the transport arising from both branches is identical. As can be seen from Eq. (57), up to dispersive corrections, the radial wave flux of parallel momentum is quenched. This result is not surprising since for the simple magnetic field topology considered, the only means for Alfvén waves to transport momentum radially is to decouple from magnetic field lines, which is only possible via the radial group velocity introduced by dispersive corrections. If we now consider the limit of finite, but weak equilibrium gradients [i.e., $|\omega_e^*/(k_\parallel v_A)| \ll 1$], to lowest order the KSAW dispersion relation is given by

$$\omega_k^\pm = \pm k_\parallel v_A \sqrt{1\pm\frac{\omega_{pi}^*}{k_\parallel v_A}\left[1+k_\perp^2\rho_s^2\left(1\pm\frac{\omega_e^*-\omega_{pi}^*}{k_\parallel v_A}\right)\right]^{1/2}}, \tag{58}$$

such that the residual stress may be written to first order in $\omega_e^*/(k_\parallel v_A)$ as

$$S_\pm = -2\sum_k (k_\perp\rho_s)^2\left(1-\frac{\tau^{-1}}{2}\mp\frac{\omega_{pi}^*}{k_\parallel v_A}\right)S_k^\pm. \tag{59}$$

Thus, the presence of finite gradients in the equilibrium profiles allow for a splitting of the positive and negative Alfvén modes, and hence a significant deviation in the functional form of $A(\hat{\beta})$ and $B(\hat{\beta})$ for the positive and negative roots. Since the splitting term is odd in $k_\parallel$, while the first two terms in Eq. (59) vanish identically after summation in the absence of $k_\parallel$ symmetry breaking, the splitting term is nonvanishing. Hence, the presence of frequency splitting allows for finite levels of transport induced by both S_k^+ or S_k^- in the absence of $k_\parallel$ symmetry breaking, with directions typically in opposition to one another. Therefore, in the presence of equilibrium profile gradients, a sufficient requirement for a nonzero residual stress is an imbalance in $|\delta\phi_k^+|^2$ versus $|\delta\phi_k^-|^2$ (i.e., an imbalance in the Elsasser populations), a different requirement than $k_\parallel$ symmetry breaking.

The functional form of the residual stress for both the ω_k^+ and ω_k^- branches of the drift-Alfvén dispersion relation is shown in Fig. 3, with the drift wave branch also plotted for reference. Since this plot is generated for a homogeneous formulation of drift-Alfvén turbulence, the structure of the radial eigenmodes, and hence $k_\parallel$ symmetry breaking, cannot be modelled comprehensively. The drift wave branch, denoted by the solid line in Fig. 3, is identically zero as anticipated, however, both the positive and negative Alfvén branches have finite amplitudes. It is also clear that their magnitudes are equal and opposite to each other, such that finite momentum transport is only induced for cases of imbalances in the underlying Elsasser population. Furthermore, as $\hat{\beta}$ is increased the direction of momentum transport induced by each branch of Alfvén turbulence inverts, and then strongly diverges. This suggests that for regimes of finite
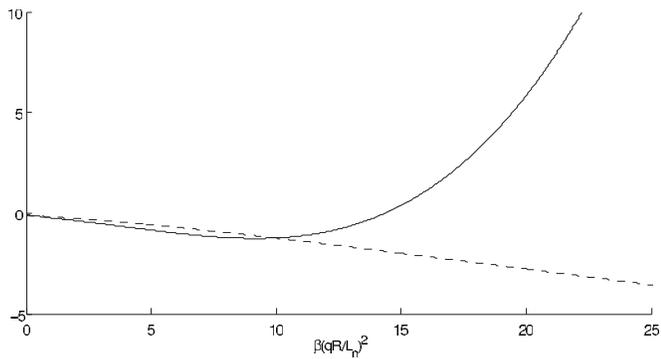
FIG. 4. Plot of positive Alfvén root demonstrating change of sign of $A(\hat{\beta})$ for high $\hat{\beta}$, with the parameters: $\tau=1$, $k_\perp\rho_i=0.3$, $\hat{\omega}_e^*=0.3/\sqrt{2}$, $\eta_i=2$, and $\hat{k}_\parallel=1.0$. The solid line corresponds to $A(\hat{\beta})$ and the broken line to $B(\hat{\beta})$.

cross helicity, robust momentum transport can be induced at high $\hat{\beta}$, as well as inversions in the direction of off-diagonal momentum transport at moderate $\hat{\beta}$.

For completeness, we have also plotted the functional forms of $A(\hat{\beta})$ and $B(\hat{\beta})$, for both branches of Alfvén turbulence (Fig. 4). As anticipated, both $A(\hat{\beta})$ and $B(\hat{\beta})$ are strong functions of $\hat{\beta}$, with $A(\hat{\beta})$ being particularly sensitive. Furthermore, from Fig. 4 the magnitude of $A(\hat{\beta})$ can be seen to be substantially larger than $B(\hat{\beta})$ for high $\hat{\beta}$. Thus, while both $A(\hat{\beta})$ and $B(\hat{\beta})$ are capable of inducing finite levels of off-diagonal transport, $A(\hat{\beta})$ is generally the dominant component. Also, since $B(\hat{\beta})$ has a definite sign, $A(\hat{\beta})$ can be seen to be responsible for introducing inversions in the direction of off-diagonal transport, where we note that the location of the inversion is highly parameter dependent.

VI. CONCLUSION

In the above analysis we have extended the derivation of electrostatic parallel momentum transport coefficients to plasmas with finite β. It is found that while the electrostatic approximation is valid for low-β plasmas throughout much of the bulk plasma region, near regions of steep density gradients the electrostatic approximation is found to provide a poor estimate of the off-diagonal momentum transport. In particular, the resonant component of parallel momentum transport, which recent gyrokinetic simulations have shown to potentially play a key role in determining the toroidal momentum diffusivity,[26] is observed to be proportional to $|\delta E_\parallel|^2$. This observation has been shown to yield the following results:

(1) For the case of drift wave turbulence, where the electrostatic and electromagnetic components of the parallel electric field oppose one another, the resonant contribution to the turbulent momentum flux (i.e., both diffusive and non-diffusive terms) can be quenched for sufficiently large $\hat{\beta} \equiv \beta(qR/L_n)^2$.

(2) For ITG turbulence on the other hand, to lowest order in $\hat{\beta}$, $\delta E_\parallel$ is enhanced leading to a potential increase in the level of transport induced by this branch of turbulence in

regions of steep pressure gradients in comparison to its electrostatic value at fixed levels of turbulence intensity.

(3) While ideal Alfvén modes have been shown to induce zero momentum transport, KSAWs are capable of introducing significant levels of momentum transport.

Qualitatively similar statements, with some amendment as discussed in detail above (see Table I), can be made for the nonresonant component of the momentum flux. More specifically, while the off-diagonal components of the momentum flux are modified in a manner fairly analogous to that for the resonant component, the diffusive contribution is not impacted by finite-$\hat{\beta}$ corrections for the modes considered.

Items (1) and (2) above, and their nonresonant analogs, can be seen to have a potentially significant impact on momentum transport in barrier regions, and thus likely on the rate of core plasma rotation. This follows, since existing theoretical models predict the residual stress contribution to the turbulent momentum flux to play a crucial role in determining the rate of toroidal rotation in the absence of external momentum input.[13] Furthermore, the residual stress is anticipated to be primarily active in regions of steep pressure gradients, such as exist near transport barriers.[7] As noted above, these regions coincide with domains in which finite-$\hat{\beta}$ effects are likely to strongly impact the turbulent momentum flux. Thus, for drift wave turbulence, the electromagnetic result indicated by item (1) suggests that an electrostatic calculation would predict an artificially large value of the residual stress term for a given spectrum $|\delta\phi_{k,\omega}|^2$ assuming a linear relation between $\delta\phi_{k,\omega}$ and $\delta A_{\parallel k,\omega}$. Furthermore, in the electrostatic limit, the only presently known means of suppressing the residual stress term is through $\mathbf{E}\times\mathbf{B}$ shear decorrelation of the underlying microturbulence. This is in contrast to the more general electromagnetic limit, where $\delta E_\parallel$ shielding via parallel induction is predicted to provide a novel means of quenching the residual stress, without extinguishing the underlying microturbulence. In contrast, to lowest order in $\hat{\beta}$, the electrostatic limit has been shown to underestimate the magnitude of the residual stress term for ITG turbulence. We note that these predictions are made assuming equivalent levels of turbulence intensity for the electrostatic and electromagnetic cases.

Kinetic Alfvén waves provide an alternate channel of momentum transport aside from the already well studied case of electrostatic microturbulence. In particular, for anticipated burning plasmas in the next generation of confinement

TABLE I. Summary of scaling trends of diffusive and nondiffusive terms with increasing $\hat{\beta}$.

	Resonant particles	Nonresonant particles
Diffusive flux:	See Eq. (53)	See Eq. (46)
Electron DWs	Reduced	Unaffected by $\hat{\beta}$ corrections
ITG	Enhanced	Unaffected by $\hat{\beta}$ corrections
Nondiffusive flux:	See Eq. (53)	See Eqs. (47)–(50)
Electron DWs	Reduced	Reduced
ITG	Enhanced	Typically enhanced

devices, the presence of a large population of energetic alpha particles can potentially lead to the destabilization of a broad spectrum of Alfvén eigenmodes. As discussed above, KSAWs may be destabilized via mode conversion of Alfvén eigenmodes such as TAEs near rational surfaces, and are thus likely to be active in present, and especially, future devices. Based on the analysis presented above, the character of the turbulent transport induced by Alfvén modes is likely to have a distinctly different nature for the cases of balanced and unbalanced Elsasser populations. In the former case, for the simple geometry utilized here, only a diffusive component to the momentum flux would be present. Thus Alfvénic turbulence would have the effect of introducing an additional diffusive contribution to the turbulent momentum flux aside from that arising from small scale microturbulence. In the latter scenario of an unbalanced Elsasser population, a residual stress component to the momentum flux would also be present, providing an additional means of inducing offsets in the plasma rotation. While a detailed analysis of mechanisms for inducing imbalances in the Elsasser population is beyond the scope of the present work, here we speculate that an unbalanced energetic particle population would likely provide such a mechanism. These and other ramifications arising from the above result will be pursued in a future publication.

ACKNOWLEDGMENTS

APPENDIX A: DERIVATION OF GYROKINETIC POISSON–AMPERE SYSTEM

Here we provide a brief derivation of the approximate gyrokinetic Poisson–Ampere system used in the above analysis. The Poisson–Ampere system may be written as[20]

$$\nabla_\perp^2 \delta\phi = -4\pi \sum_s q_s \int d^3\bar{v} \langle e^{-\rho\cdot\nabla} T_{\mathrm{gy}} F^{(s)} \rangle_\alpha, \quad (A1a)$$

$$\nabla_\perp^2 \delta A_\parallel = -\frac{4\pi}{c} \sum_s q_s \int d^3\bar{v} \langle e^{-\rho\cdot\nabla} v_\parallel T_{\mathrm{gy}} F^{(s)} \rangle_\alpha, \quad (A1b)$$

where the charge density and current density are written in terms of the gyrocenter distribution function. The pullback transformation can be shown to be given by

$$T_{\mathrm{gy}} F^{(s)} = F^{(s)} + [S_1, F^{(s)}] + \frac{q_s}{c}(\delta A_{\parallel\mathrm{gc}} - \langle \delta A_{\parallel\mathrm{gc}} \rangle_\alpha) \frac{\partial F^{(s)}}{\partial v_\parallel}, \quad (A2)$$

where

$$S_1 \equiv \frac{q_s}{\omega_{cs}} \int^\alpha d\alpha (\delta\psi_{\mathrm{gc}} - \langle \delta\psi_{\mathrm{gc}} \rangle_\alpha),$$

$$\delta\psi_{\mathrm{gc}} = \delta\phi_{\mathrm{gc}} - \frac{v_\parallel}{c}(\delta A_{\parallel\mathrm{gc}} - \langle \delta A_{\parallel\mathrm{gc}} \rangle_\alpha), \quad \delta\xi_{\mathrm{gc}} = e^{\rho\cdot\nabla}\delta\xi,$$

and the gyrokinetic Poisson bracket can be shown to reduce to

$$[S_1, F^{(s)}] = \frac{q_s}{m_s c} \frac{\partial S_1}{\partial\alpha} \frac{\partial\langle F^{(s)}\rangle}{\partial\mu} + \frac{\hat{b}}{m_s}\cdot\nabla S_1 \frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel}$$
$$- \frac{c\hat{b}}{q_s B}\cdot[\nabla S_1 \times \nabla\langle F^{(s)}\rangle]. \quad (A3)$$

Ordering the ratio of the first term in Eq. (A3) versus the second and third terms, yields respectively,

$$\frac{q_s}{m_s c}\frac{1}{\mu}\frac{m_s}{k_\parallel} v_{ths} \sim \frac{1}{k_\parallel \rho_{(s)}} \gg 1,$$

$$\frac{q_s}{m_s c}\frac{q_s B}{c}\frac{1}{\mu}\frac{L_\perp}{k_y} \sim \left[\frac{L_\perp}{\rho_{(s)}}\right]\left[\frac{1}{k_y \rho_{(s)}}\right] \gg 1.$$

Thus Eq. (A2) can be simplified as

$$T_{\mathrm{gy}} F^{(s)} = F^{(s)} + \frac{q_s}{m_s c}\frac{\partial\langle F^{(s)}\rangle}{\partial\mu}\frac{\partial S_1}{\partial\alpha} + \frac{q_s}{c}(\delta A_{\parallel\mathrm{gc}} - \langle \delta A_{\parallel\mathrm{gc}} \rangle_\alpha)\frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel}. \quad (A4)$$

For a Maxwellian this expression may be rewritten as

$$T_{\mathrm{gy}} F^{(s)} = F^{(s)} + \frac{q_s}{T_s}(\delta\phi_{\mathrm{gc}} - \langle \delta\phi_{\mathrm{gc}} \rangle_\alpha)\langle F^{(s)}\rangle$$
$$+ \frac{q_s}{T_s}\frac{v_\parallel}{c}(\delta A_{\parallel\mathrm{gc}} - \langle \delta A_{\parallel\mathrm{gc}} \rangle_\alpha)\langle F^{(s)}\rangle$$
$$+ \frac{q_s}{c}(\delta A_{\parallel\mathrm{gc}} - \langle \delta A_{\parallel\mathrm{gc}} \rangle_\alpha)\frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel}, \quad (A5)$$

where we note that for a centered Maxwellian the third and fourth terms in Eq. (A5) cancel. Here, however, since we are interested in generalized velocity profiles, we consider a shifted Maxwellian, such that Eq. (A5) reduces to

$$T_{\mathrm{gy}} F^{(s)} = F^{(s)} + \frac{q_s}{T_s}(\delta\phi_{\mathrm{gc}} - \langle \delta\phi_{\mathrm{gc}} \rangle_\alpha)\langle F^{(s)}\rangle$$
$$+ \frac{q_s}{T_s}\frac{\langle v_\parallel\rangle}{c}(\delta A_{\parallel\mathrm{gc}} - \langle \delta A_{\parallel\mathrm{gc}} \rangle_\alpha)\langle F^{(s)}\rangle. \quad (A6)$$

The presence of the third term in Eq. (A6) will be seen to play a key role in the structure of gyrokinetic Poisson–Ampere system. Substituting Eq. (A6) into Eqs. (A1a) and (A1b) yields the gyrokinetic Poisson–Ampere system,

$$k_\perp^2 \delta\phi = 4\pi \sum_s q_s \int d^3\bar{v} \left\{ J_0(\lambda) \delta F_{k,\omega}^{(s)} \right.$$
$$\left. + [J_0^2(\lambda) - 1]\frac{q_s}{T_s}\langle F^{(s)}\rangle\left(\delta\phi_{k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta A_{\parallel k,\omega}\right) \right\}, \quad (A7a)$$

$$k_\perp^2 \delta A_{\parallel k,\omega} = \frac{4\pi}{c}\sum_s q_s \int d^3\bar{v}v_\parallel \left\{ J_0(\lambda)\delta F_{k,\omega}^{(s)} \right.$$
$$\left. + [J_0^2(\lambda) - 1]\frac{q_s}{T_s}\langle F^{(s)}\rangle\left(\delta\phi_{k,\omega} - \frac{\langle v_\parallel\rangle}{c}\delta A_{\parallel k,\omega} \right) \right\},$$

$$\text{(A7b)}$$

which completes the derivation.

APPENDIX B: DIELECTRIC OF GYROKINETIC MEDIA

1. Derivation of susceptibility tensor

In order to derive a compact expression for the gyrokinetic susceptibility tensor, it is useful to utilize the covariant notation introduced in Sec. II. To first order in the fluctuating fields, the gyrokinetic equation may be written as

$$0 = \partial^\alpha[v_\alpha F^{(s)}] + \frac{\partial}{\partial \mathbf{x}_\perp}\cdot[\dot{\mathbf{X}}_\perp F^{(s)}] + \frac{\partial}{\partial v_\parallel}[\dot{V}_\parallel F^{(s)}], \quad \text{(B1)}$$

where

$$\dot{\mathbf{X}}_\perp = -J_0(\lambda)\frac{1}{B}(\hat{\mathbf{b}}\times\nabla_\perp)v^\alpha\psi_\alpha, \quad \text{(B2)}$$

$$\dot{V}_\parallel = -J_0(\lambda)\frac{q_s}{m_s}(\partial^1\psi^0 - \partial^0\psi^1) = \frac{1}{2}J_0(\lambda)\frac{q_s}{m_s}\epsilon_{\alpha\beta}F^{\alpha\beta}, \quad \text{(B3)}$$

and

$$\lambda \equiv k_\perp\rho_\perp z^\alpha \equiv \begin{pmatrix} ct \\ z \end{pmatrix}, \quad v^\alpha \equiv \dot{z}^\alpha = \begin{pmatrix} c \\ v_\parallel \end{pmatrix},$$

$$\partial^\alpha \equiv \frac{\partial}{\partial z_\alpha} = \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial z} \end{pmatrix}, \quad F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_\parallel \\ E_\parallel & 0 \end{pmatrix},$$

$$E_\parallel = -(\partial^1\psi^0 - \partial^0\psi^1).$$

From Eq. (B1) the induced plasma response may be written

$$\delta F_{k,\omega}^{(s)} = J_0(\lambda)\frac{c}{B}R(k_\alpha v^\alpha)\left[k_y\frac{v_\gamma}{c}\left(\frac{\partial}{\partial x} - \langle v_\parallel\rangle'\frac{\partial}{\partial v_\parallel} \right)\langle F^{(s)}\rangle \right.$$
$$\left. - \frac{1}{2}\omega_{cs}\epsilon_{\alpha\beta}G_\gamma^{\alpha\beta}\frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel} \right]\psi^\gamma \equiv T_\gamma^{(s)}\psi^\gamma, \quad \text{(B4)}$$

where we have transformed to a frame moving with velocity $\langle v_\parallel\rangle$, and introduced the definitions,

$$k^\alpha \equiv \begin{pmatrix} \omega/c \\ k_\parallel \end{pmatrix}, \quad G_\gamma^{\alpha\beta} \equiv k^\alpha\delta_\gamma^\beta - k^\beta\delta_\gamma^\alpha, \quad R(k_\alpha v^\alpha) \equiv \frac{1}{\omega - k_\parallel v_\parallel}.$$

Using the above notation it is now straightforward to calculate the gyrokinetic dielectric. From Eq. (B4) the induced plasma response may be written in terms of total plasma perturbation (including both the induced field as well as the external field) as

$$\delta F_{k,\omega}^{\text{ind}} = T_\gamma^{(s)}(\psi^{\text{tot}})^\gamma, \quad \text{(B5)}$$

where $(\psi^{\text{tot}})^\gamma = (\psi^{\text{ind}})^\gamma + (\psi^{\text{ext}})^\gamma$. The current two vector may be defined in terms of the gyrokinetic response, i.e.,

$$(j^{\text{ind}})^\delta \equiv \sum_s q_s \int d^3\bar{v}J_0(\lambda)v^\delta\delta F_{k,\omega}^{\text{ind}}. \quad \text{(B6)}$$

Substituting Eq. (B5) into Eq. (B6), yields

$$(j^{\text{ind}})^\delta \equiv \sum_s q_s \int d^3\bar{v}J_0(\lambda)v^\delta T_\gamma^{(s)}(\psi^{\text{tot}})^\gamma \equiv \mathrm{R}_\gamma^\delta(\psi^{\text{tot}})^\gamma. \quad \text{(B7)}$$

Thus, inserting Eq. (B7) into Eq. (6) allows the induced response of the fields to an external perturbations to be written as

$$\left(M_\gamma^\delta - \frac{4\pi}{c}R_\gamma^\delta \right)(\psi^{\text{ind}})^\gamma = \frac{4\pi}{c}R_\gamma^\delta(\psi^{\text{ext}})^\gamma. \quad \text{(B8)}$$

Multiplying Eq. (B8) by $(M^{-1})_\gamma^\sigma$, yields,

$$\left[\delta_\gamma^\sigma - \frac{4\pi}{c}(M^{-1})_\delta^\sigma R_\gamma^\delta \right](\psi^{\text{ind}})^\gamma = \frac{4\pi}{c}(M^{-1})_\delta^\sigma R_\gamma^\sigma(\psi^{\text{ext}})^\gamma. \quad \text{(B9)}$$

The susceptibility tensor may then be defined as

$$\chi_\gamma^\sigma \equiv -\frac{4\pi}{c}(M^{-1})_\delta^\sigma R_\gamma^\delta. \quad \text{(B10)}$$

2. Explicit form of susceptibility tensor

Introducing the simplified notation,

$$\chi_{k,\omega}^{\phi\phi} = \chi_0^0, \quad \chi_{k,\omega}^{\phi A} = \chi_1^0, \quad \chi_{k,\omega}^{A\phi} = \chi_0^1, \quad \chi_{k,\omega}^{AA} = \chi_1^1, \quad \text{(B11)}$$

the components of the susceptibility matrix may be defined as

$$\chi_{k,\omega}^{\phi\phi} = -4\pi\sum_s \frac{q_s}{\epsilon_\perp(k)k_\perp^2}\int d^3\bar{v}\frac{J_0^2[k_\perp\rho_\perp^{(s)}]}{\omega - k_\parallel v_\parallel}$$
$$\times\left[\frac{c}{B}(\hat{\mathbf{b}}\times\mathbf{k})\cdot\left(\frac{\partial}{\partial\mathbf{x}} - \frac{\partial\langle v_\parallel\rangle}{\partial\mathbf{x}}\frac{\partial}{\partial v_\parallel} \right)\langle F^{(s)}\rangle - \frac{q_s}{m_s}k_\parallel\frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel} \right],$$

$$\text{(B12a)}$$

$$\chi_{k,\omega}^{\phi A} = 4\pi\sum_s \frac{q_s}{\epsilon_\perp(k)k_\perp^2}\int d^3\bar{v}\frac{J_0^2[k_\perp\rho_\perp^{(s)}]}{\omega - k_\parallel v_\parallel}$$
$$\times\left[\frac{v_\parallel}{B}(\hat{\mathbf{b}}\times\mathbf{k})\cdot\left(\frac{\partial}{\partial\mathbf{x}} - \frac{\partial\langle v_\parallel\rangle}{\partial\mathbf{x}}\frac{\partial}{\partial v_\parallel} \right)\langle F^{(s)}\rangle - \frac{q_s}{m_s}\frac{\omega}{c}\frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel} \right],$$

$$\text{(B12b)}$$

$$\chi_{k,\omega}^{A\phi} = -\frac{4\pi}{c}\sum_s \frac{q_s}{k_\perp^2}\int d^3\bar{v}v_\parallel\frac{J_0^2[k_\perp\rho_\perp^{(s)}]}{\omega - k_\parallel v_\parallel}$$
$$\times\left[\frac{c}{B}(\hat{\mathbf{b}}\times\mathbf{k})\cdot\left(\frac{\partial}{\partial\mathbf{x}} - \frac{\partial\langle v_\parallel\rangle}{\partial\mathbf{x}}\frac{\partial}{\partial v_\parallel} \right)\langle F^{(s)}\rangle - \frac{q_s}{m_s}k_\parallel\frac{\partial\langle F^{(s)}\rangle}{\partial v_\parallel} \right],$$

$$\text{(B12c)}$$

$$\chi_{k,\omega}^{AA} = \frac{4\pi}{c} \sum_s \frac{q_s}{k_\perp^2} \int d^3\bar{v}\, v_\| \frac{J_0^2[k_\perp \rho_\perp^{(s)}]}{\omega - k_\| v_\|}$$

$$\times \left[\frac{v_\|}{B}(\hat{\mathbf{b}} \times \mathbf{k}) \cdot \left(\frac{\partial}{\partial \mathbf{x}} - \frac{\partial \langle v_\| \rangle}{\partial \mathbf{x}} \frac{\partial}{\partial v_\|} \right) \langle F^{(s)} \rangle - \frac{q_s}{m_s} \frac{\omega}{c} \frac{\partial \langle F^{(s)} \rangle}{\partial v_\|} \right].$$

(B12d)

By adding and subtracting ω to the numerators of Eqs. (B12c) and (B12d), the following expressions may be derived:

$$\chi_{k,\omega}^{AA} = \epsilon_\perp(k) \frac{\omega}{ck_\|} \chi_{k,\omega}^{\phi A} - \sum_s \frac{k_{Ds}^2}{k_\perp^2} \frac{v_{ths}^2}{c^2} \frac{k_y}{k_\|} \Gamma_0[b^{(s)}] \frac{1}{\omega_{cs}} \frac{\partial \langle v_\| \rangle}{\partial x}, \quad (B13a)$$

$$\chi_{k,\omega}^{A\phi} = \epsilon_\perp(k) \frac{\omega}{ck_\|} \chi_{k,\omega}^{\phi\phi} - \sum_s \frac{k_{Ds}^2}{k_\perp^2} \frac{k_y}{k_\|} \frac{v_{ths}}{c}$$

$$\times \rho^{(s)} \Gamma_0[b^{(s)}] \left(\frac{\partial}{\partial x} \ln n_s - b^{(s)} \left\{ 1 - \frac{\Gamma_1[b^{(s)}]}{\Gamma_0[b^{(s)}]} \right\} \frac{\partial}{\partial x} \ln T_s \right).$$

(B13b)

Explicitly, $\chi_{k,\omega}^{\phi A}$ and $\chi_{k,\omega}^{\phi\phi}$ can written as

$$\chi_{k,\omega}^{\phi\phi} = \sum_s \frac{-1}{\epsilon_\perp(k)} \frac{k_{Ds}^2}{k_\perp^2} \frac{k_y}{k_\|}$$

$$\times \Gamma_0(b) \left[\frac{1}{\sqrt{2}} \left(-\frac{\rho^{(s)}}{L_n} + \left\{ \frac{1}{2} + b \left[1 - \frac{I_1(b)}{I_0(b)} \right] \right\} \frac{\rho^{(s)}}{L_{Ts}} \right) Z(\zeta_s) \right.$$

$$+ \left[\frac{1}{\omega_{cs}} \frac{\partial \langle v_\| \rangle}{\partial x} - \frac{k_\|}{k_y} - \frac{1}{2} \frac{\rho^{(s)}}{L_{Ts}} \frac{\omega}{v_{ths}k_\|} \right]$$

$$\times \left. \left[1 + \frac{1}{\sqrt{2}} \frac{\omega}{v_{ths}k_\|} Z(\zeta_s) \right] \right],$$

(B14a)

$$\chi_{k,\omega}^{\phi A} = \sum_s \frac{1}{\epsilon_\perp(k)} \frac{k_{Ds}^2}{k_\perp^2} \frac{k_y}{k_\|} \frac{v_{ths}}{c}$$

$$\times \Gamma_o(b) \left[-\frac{1}{2} \frac{\rho^{(s)}}{L_{Ts}} + \left(-\frac{\rho^{(s)}}{L_n} + \left\{ \frac{1}{2} + b \left[1 - \frac{I_1(b)}{I_0(b)} \right] \right. \right. \right.$$

$$\left. \left. - \frac{1}{2} \frac{\omega^2}{v_{ths}^2 k_\|^2} \right\} \frac{\rho^{(s)}}{L_{Ts}} - \frac{1}{k_y \rho^{(s)}} \frac{\omega}{\omega_{cs}} + \frac{\omega}{v_{ths}k_\|} \frac{1}{\omega_{cs}} \frac{\partial \langle v_\| \rangle}{\partial x} \right)$$

$$\times \left. \left[1 + \frac{1}{\sqrt{2}} \frac{\omega}{v_{ths}k_\|} Z(\zeta_s) \right] \right].$$

(B14b)

In order to further simplify Eqs. (B13a), (B13b), (B14a), and (B14b), it is useful to take the simple limit of $m_e \to 0$ and $|k_\| v_{thi}/\omega| < 1$, yielding for the real piece,

$$\text{Re } \chi_{k,\omega}^{\phi\phi} \approx \frac{1}{\epsilon_\perp(k)} \frac{k_{De}^2}{k_\perp^2} \left[1 - \frac{\omega_e^*}{\omega} \Gamma_0(b) \left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c} \right) \right]$$

$$- \frac{\Gamma_0(b)}{\epsilon_\perp(k)} \frac{k_{De}^2}{k_\perp^2} \left(\frac{v_{thi}k_\|}{\omega} \right)^2 \left[\frac{\omega_e^*}{\omega} \left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c} \right) \right.$$

$$\left. + \tau \left(1 - \frac{k_y}{k_\|} \frac{1}{\omega_{ci}} \frac{\partial \langle v_\| \rangle}{\partial x} \right) \right],$$

(B15)

$$\text{Re } \chi_{k,\omega}^{\phi A} \approx -\frac{1}{\epsilon_\perp(k)} \frac{k_{De}^2}{k_\perp^2} \left(\frac{\omega}{ck_\|} \right) \left(1 - \frac{\omega_e^*}{\omega} \right) + \frac{\Gamma_0(b)}{\epsilon_\perp(k)} \frac{k_{De}^2}{k_\perp^2} \left(\frac{\omega}{ck_\|} \right)$$

$$\times \left(\frac{v_{thi}k_\|}{\omega} \right)^2 \left[\frac{\omega_e^*}{\omega} \left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c} \right) \right.$$

$$\left. + \tau \left(1 - \frac{k_y}{k_\|} \frac{1}{\omega_{ci}} \frac{\partial \langle v_\| \rangle}{\partial x} \right) \right],$$

(B16)

where ω is taken as pure real. From Eqs. (B13a) and (B13b), the real parts of $\chi_{k,\omega}^{AA}$ and $\chi_{k,\omega}^{A\phi}$ can be written as

$$\text{Re } \chi_{k,\omega}^{AA} \approx -\frac{1}{k_\perp^2 \rho_s^2} \left(\frac{\omega}{v_A k_\|} \right)^2 \left(1 - \frac{\omega_e^*}{\omega} \right)$$

$$+ \frac{\Gamma_0(b)}{k_\perp^2 \rho_s^2} \left(\frac{\omega}{v_A k_\|} \right)^2 \left(\frac{v_{thi}k_\|}{\omega} \right)^2$$

$$\times \left[\frac{\omega_e^*}{\omega} \left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c} \right) + \tau \left(1 - \frac{k_y}{k_\|} \frac{1}{\omega_{ci}} \frac{\partial \langle v_\| \rangle}{\partial x} \right) \right],$$

(B17)

$$\text{Re } \chi_{k,\omega}^{A\phi} \approx \frac{c}{v_A} \frac{1}{k_\perp^2 \rho_s^2} \left(\frac{\omega}{v_A k_\|} \right) \left(1 - \frac{\omega_e^*}{\omega} \right) - \frac{\Gamma_0(b)}{k_\perp^2 \rho_s^2} \frac{c}{v_A} \left(\frac{\omega}{v_A k_\|} \right)$$

$$\times \left(\frac{v_{thi}k_\|}{\omega} \right)^2$$

$$\times \left[\frac{\omega_e^*}{\omega} \left(1 + \frac{3}{2}\eta_i - \frac{\eta_i}{\eta_c} \right) + \tau \left(1 - \frac{k_y}{k_\|} \frac{1}{\omega_{ci}} \frac{\partial \langle v_\| \rangle}{\partial x} \right) \right],$$

(B18)

where terms proportional to $k_\perp^2 \rho_i^2 (v_{thi}k_\|/\omega)^2$ have been dropped. Here it is useful to note the relationship

$$-\left(\frac{\omega}{ck_\|} \right) \text{Re } \chi_{k,\omega}^{A\phi} = \text{Re } \chi_{k,\omega}^{AA}.$$

(B19)

Similarly for the imaginary components, where we only keep the lowest order contribution in $(v_{thi}k_\|/\omega)^2$, and neglect resonant particle contributions for simplicity,

$$\text{Im } \chi_{k,\omega}^{\phi\phi} = \frac{\gamma}{\omega} \frac{\Gamma_0(b)}{\epsilon_\perp(k)} \frac{k_{De}^2}{k_\perp^2} \frac{\omega_e^*}{\omega} \left(1 + \frac{\eta_i}{2} - \frac{\eta_i}{\eta_c} \right),$$

(B20a)

$$\text{Im } \chi_{k,\omega}^{\phi A} = -\frac{\gamma}{\omega} \frac{1}{\epsilon_\perp(k)} \frac{k_{De}^2}{k_\perp^2} \left(\frac{\omega}{ck_\|} \right),$$

(B20b)

$$\text{Im } \chi_{k,\omega}^{AA} = -\frac{\gamma}{\omega} \frac{1}{k_\perp^2 \rho_s^2} \left(\frac{\omega}{v_A k_\|} \right)^2 \left(2 - \frac{\omega_e^*}{\omega} \right),$$

(B20c)

$$\text{Im } \chi_{k,\omega}^{A\phi} = \frac{\gamma}{\omega} \frac{1}{k_\perp^2 \rho_s^2} \frac{c}{v_A} \left(\frac{\omega}{v_A k_\|} \right).$$

(B20d)

APPENDIX C: LINEAR MODE PROPERTIES

The momentum theorem derived in Sec. III provides a specific form for calculating the radial flux of parallel momentum. Here it is convenient to derive the linear mode properties for general gyrokinetic modes, as well as eigen-

vectors which connect the scalar potential with the parallel component of the vector potential such that expressions for parallel momentum flux can be readily analyzed in terms of a single field.

1. Homogeneous turbulence

In this subsection we outline the structure of the dispersion relationship and eigenvectors for electromagnetic gyrokinetic modes. The parallel component of the vector potential may be related to the scalar potential via Eq. (B9), yielding

$$\delta A_{\parallel k,\omega} = \frac{-\chi_{k,\omega}^{A\phi}}{1 + \chi_{k,\omega}^{AA}} \delta\phi_{k,\omega}, \tag{C1}$$

where $\chi_{k,\omega}^{A\phi}$ and $\chi_{k,\omega}^{AA}$ are defined by Eqs. (B13a), (B13b), (B14a), and (B14b). Similarly, a dispersion relation may be derived for electromagnetic gyrokinetic modes, yielding

$$0 = D_{k,\omega} = (1 + \chi_{k,\omega}^{\phi\phi})(1 + \chi_{k,\omega}^{AA}) - \chi_{k,\omega}^{\phi A}\chi_{k,\omega}^{A\phi}. \tag{C2}$$

From Eq. (C2), the growth rate may be approximated as

$$\gamma_k = \left. \frac{-\operatorname{Im} D_{k,\omega}}{\partial D_{k,\omega}/\partial\omega} \right|_{\omega=\omega_k}, \tag{C3}$$

where

$$\operatorname{Im} D_k = \operatorname{Im}\chi_{k,\omega}^{\phi\phi} - \frac{\chi_{k,\omega}^{A\phi}}{1 + \chi_{k,\omega}^{AA}} \operatorname{Im}\chi_{k,\omega}^{\phi A}. \tag{C4}$$

2. Sheared slab geometry

The linear eigenmode equations for electromagnetic gyrokinetic modes in a sheared slab geometry can be written utilizing the susceptibility matrix as

$$-\epsilon_\perp(k)k_\perp^2\,\delta\phi_k = \epsilon_\perp(k)k_\perp^2(\chi_k^{\phi\phi}\delta\phi_k + \chi_k^{\phi A}\delta A_{\parallel k}), \tag{C5a}$$

$$-k_\perp^2\,\delta A_{\parallel k} = k_\perp^2(\chi_k^{A\phi}\delta\phi_k + \chi_k^{AA}\delta A_{\parallel k}), \tag{C5b}$$

where $k_\perp^2 = -\partial^2/\partial x^2 + k_y^2$, $k_\parallel = k_y x/L_s$ and the susceptibilities are defined in Appendix B. These expressions may be simplified in the fluid limit by expanding the plasma dispersion function for $\omega_k/(v_{thi}k_\parallel) > 1$, yielding the reduced equations

$$\rho_s^2\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\delta\phi_k$$
$$= \left[\alpha^{-1}\left(1 - \frac{\omega_e^*}{\bar\omega_k}\right) - \frac{1}{\alpha}\frac{\omega_e^*}{\bar\omega_k^2}\left(\sigma\frac{\partial\langle v_E\rangle}{\partial x} - \frac{L_n}{L_s}\frac{\partial\langle v_\parallel\rangle}{\partial x}\right)k_y x\right.$$
$$\left. - \left(\frac{c_s k_y}{\bar\omega_k}\right)^2\frac{x^2}{L_s^2}\right]$$
$$\times\left[\delta\phi_k - \left(\frac{\bar\omega_k}{ck_y}\right)\frac{L_s}{x}\left(1 - k_y x\frac{1}{\bar\omega_k}\frac{\partial\langle v_E\rangle}{\partial x}\right)\delta A_{\parallel k}\right], \tag{C6a}$$

$$\frac{x}{L_s}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\delta A_{\parallel k} = \alpha\frac{c}{v_A}\left(\frac{\bar\omega_k}{v_A k_y}\right)\left(1 - \kappa\frac{x}{v_A}\frac{\partial\langle v_E\rangle}{\partial x}\right)$$
$$\times\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\delta\phi_k, \tag{C6b}$$

where we have defined

$$\alpha \equiv \left(1 - \frac{\omega_{pi}^*}{\bar\omega_k}\right), \quad \kappa \equiv 1 + \frac{\tau}{\alpha}\frac{\omega_{pi}^*}{\bar\omega_k}\frac{k_y v_A}{\bar\omega_k},$$

$$\sigma \equiv 1 + \left(1 - \frac{\omega_{pi}^*}{\bar\omega_k}\right)^{-1}\left(1 - \frac{\omega_e^*}{\bar\omega_k}\right)\left(\frac{1 + \eta_i}{\tau}\right),$$

and $\bar\omega_k \equiv \omega_k - k_y\langle v_E(0)\rangle$, with an $\mathbf{E}\times\mathbf{B}$ shear profile given by $\langle v_E(x)\rangle = \langle v_E(0)\rangle + \langle v_E(0)\rangle' x$. In the electrostatic limit, Eq. (C6a) can be seen to reduce to that studied in Ref. 27 with the addition of $\mathbf{E}\times\mathbf{B}$ flow shear. In this simple limit, this eigenvalue equation can be shown to possess three roots: stable and unstable ITG roots, and a marginally stable drift wave branch. The presence of magnetic fluctuations allows for two additional roots given by parallel and antiparallel propagating Alfvén waves. Exact analytic solutions to the generalized electromagnetic system given by Eqs. (C6a) and (C6b) are at present unavailable (although see Ref. 28 for limiting cases) such that we will only be able to determine approximate asymptotic forms. Introducing eikonal solutions of the form,[29,30]

$$[\delta\phi_k(x), \delta A_{\parallel k}(x)] = [\Phi(x), A(x)]\exp\left[i\int^x dx\,k_x(x)\right], \tag{C7}$$

where (Φ, A) are assumed to be slow functions of x. After substitution of Eq. (C7) into Eqs. (C6a) and (C6b), the lowest order eigenvalue equations have the form,

$$k_x^2\rho_s^2\Phi = -k_y^2\rho_s^2\Phi + \left[\left(\frac{c_s k_y}{\bar\omega_k}\right)^2\left(\frac{x - x_0}{L_s}\right)^2 - \frac{1}{\alpha}\left(1 - \frac{\omega_e^*}{\bar\omega_k}\right)\right]$$
$$\times\left[\Phi - \left(\frac{\bar\omega_k}{ck_y}\right)\frac{L_s}{x}\left(1 - k_y x\frac{1}{\bar\omega_k}\frac{\partial\langle v_E\rangle}{\partial x}\right)A\right], \tag{C8a}$$

$$\frac{x}{L_s}(k_x^2 + k_y^2)A = \alpha\frac{c}{v_A}\left(\frac{\bar\omega_k}{v_A k_y}\right)\left(1 - \frac{\kappa x}{v_A}\frac{\partial\langle v_E\rangle}{\partial x}\right)(k_x^2 + k_y^2)\Phi, \tag{C8b}$$

where

$$x_0 \equiv -\frac{1}{2}L_s\frac{1}{\alpha}\frac{L_s}{L_n}\frac{1}{\omega_{ci}}\left(\sigma\frac{\partial\langle v_E\rangle}{\partial x} - \frac{L_n}{L_s}\frac{\partial\langle v_\parallel\rangle}{\partial x}\right).$$

Here we have assumed that the $\mathbf{E}\times\mathbf{B}$ and toroidal flow shear are sufficiently weak such that we may ignore terms of order

x_0^2/L_s^2, and we have neglected spatial derivatives of Φ and A. A trivial solution of Eq. (C8b) can be easily seen to be given by $k_x^2 = -k_y^2$. The eigenvector for this solution follows from Eq. (C8a), and is given by

$$\Phi = \left(\frac{\bar{\omega}_k}{ck_y}\right)\frac{L_s}{x}\left(1 - k_y x\frac{1}{\bar{\omega}_k}\frac{\partial\langle v_E\rangle}{\partial x}\right)A. \tag{C9}$$

Neglecting $\mathbf{E}\times\mathbf{B}$ shear, Eq. (C9) can be rewritten as $\Phi = (\bar{\omega}_k/ck_\parallel)A$, which can be easily recognized as the lowest order eigenvector for Alfvén waves. Thus, asymptotic solutions with the above eigenvector, and the asymptotic radial eigenmode given by $\exp(-|k_y x|)$ (where the minus/plus solution is chosen for positive/negative k_y since we require solutions which vanish at $\pm\infty$) can be confidently identified as describing the two Alfvén roots of Eqs. (C6a) and (C6b). Two characteristics of this solution are worthy of note: Unlike electrostatic drift wave microturbulence, the presence of $\mathbf{E}\times\mathbf{B}$ flow shear does not shift the eigenmode off the resonant surface. Also, while these modes remain centered around their respective rational surfaces, $\mathbf{E}\times\mathbf{B}$ flow shear does induce a shift in the eigenvector. Physically, this shift arises due to the inductive component of the parallel electric field being Doppler shifted by the $\mathbf{E}\times\mathbf{B}$ flow. For a sheared $\mathbf{E}\times\mathbf{B}$ flow profile, the Doppler shift will introduce a component of the parallel electric field which has odd parity about the rational surface, and thus can be understood to provide an effective symmetry breaking mechanism.

The second solution to Eqs. (C8a) and (C8b), which will describe the asymptotic form of the drift wave and ITG branches of Eqs. (C6a) and (C6b), can be approximated in the limit of large x as

$$k_x^2\rho_i^2 \approx \left(\frac{v_{thi}k_y}{\bar{\omega}_k}\right)^2\left(\frac{x-x_0}{L_s}\right)^2, \tag{C10a}$$

$$A = \alpha\frac{c}{v_A}\left(\frac{\bar{\omega}_k}{v_A k_y}\right)\left(1 - \kappa\frac{\langle v_E\rangle' x}{v_A}\right)\frac{L_s}{x}\Phi. \tag{C10b}$$

Thus, this solution has a radial eigenmode of the form $\exp[-(i/2)(v_{thi}|k_y|/\bar{\omega}_k)(L_s/\rho_i)(x-x_0)^2/L_s^2]$, where we have imposed outgoing wave boundary conditions such that the negative root is selected.[31] Similar to the electrostatic case, both $\mathbf{E}\times\mathbf{B}$ and toroidal flow shear introduce a shift of the eigenmode away from its rational surface.

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